Jacobians in the photometric tracking

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1 introduction

The paper will explain the Jacobian calculations used in our photometric tracking (RGB tracker). The RGB tracker seeks to obtain the $SE(3)$ transformation $\boldsymbol{T}_{WL}$, the pose of live frame with respect to the world coordinate, that aligns the live frame to the world frame. It works by rendering the reference frame, $\underrightarrow{\mathcal{F}}_R$, into the live frame, $\underrightarrow{\mathcal{F}}_L$, using the depth rendered from the reconstructed model on the reference frame ${}_R\mathbf{D}$, and then minimizing the photometric residual.

2 Notations

The following notations will be used in the remaining parts.

${}_A\mathbf{I}_B$	Image frame originally of $\underrightarrow{\mathcal{F}}_B$ expressed in the frame $\underrightarrow{\mathcal{F}}_A$
${}_R\mathbf{D}$	Depth on the reference frame $\underrightarrow{\mathcal{F}}_R$, rendered from the model
${}_A\mathbf{V}$	3D world vertex position with respect to the frame $\underrightarrow{\mathcal{F}}_A$
${}_A\mathbf{u}$	2D pixel position represented on the frame $\underrightarrow{\mathcal{F}}_A$
$\boldsymbol{T}_{AB}$	Pose of the frame $\underrightarrow{\mathcal{F}}_B$ with respect to the frame $\underrightarrow{\mathcal{F}}_A$
π	Perspective projection

3 RGB tracker

The cost function for the photometric tracking would be:

$$
\begin{aligned}
e_{RGB}(0) &= {}_L\mathbf{I}_L - {}_R\mathbf{I}_L && (1)\\
&= {}_R\mathbf{I}_R({}_R\mathbf{u}) - {}_R\mathbf{I}_L(\pi(\boldsymbol{T}_{LR}\,(\pi^{-1}({}_R\mathbf{u}, {}_R\mathbf{D})))) && (2)\\
&= {}_R\mathbf{I}_R({}_R\mathbf{u}) - {}_R\mathbf{I}_L(\pi(\boldsymbol{T}_{WL}^{-1}\,(\boldsymbol{T}_{WR}\,({}_R\mathbf{u}, {}_R\mathbf{D})))) && (3)
\end{aligned}
$$

Some intermediate terms in the Eq. 3 have their physical meaning in this image rendering process and can be denoted as:

$$
e_{RGB}(0) = {}_R\mathbf{I}_R({}_R\mathbf{u}) - {}_R\mathbf{I}_L\,(\underbrace{\pi\,(\overbrace{\boldsymbol{T}_{WL}^{-1}\;(\underbrace{\boldsymbol{T}_{WR}\,({}_R\mathbf{u}, {}_R\mathbf{D})}_{{}_W\mathbf{V}})}^{{}_L\mathbf{V}})}_{{}_L\mathbf{u}}) \qquad (4)
$$

With small perturbation $\delta\boldsymbol{\xi}$ performed on the $\boldsymbol{T}_{WL}$ we will obtain the following equation:

$$
e_{RGB}(\delta\boldsymbol{\xi}) = {}_R\mathbf{I}_R({}_R\mathbf{u}) - {}_R\mathbf{I}_L(\pi(\boldsymbol{T}_{WL}^{-1}\,\exp((-\delta\boldsymbol{\xi})^{\wedge})_W\mathbf{V})) \qquad (5)
$$

Thus, we can get the Jacobian formula as:

$$
\begin{aligned}
J &= \frac{\partial(e_{RGB}(\delta\boldsymbol{\xi}) - e_{RGB}(0))}{\partial(\delta\boldsymbol{\xi})} && (6)\\
&= -\frac{\partial\left({}_R\mathbf{I}_L(\pi(\boldsymbol{T}_{WL}^{-1}\,\exp((-\delta\boldsymbol{\xi})^{\wedge})_W\mathbf{V}) - {}_R\mathbf{I}_L(\pi(\boldsymbol{T}_{WL}^{-1}\,({}_W\mathbf{V})\right)}{\partial(\delta\boldsymbol{\xi})} && (7)
\end{aligned}
$$

Using the chain rule, we can decompose Eq. 7 as three parts:

$$
J = -\frac{\partial\left({}_R\mathbf{I}_L({}_L\mathbf{u})\right)}{\partial({}_L\mathbf{u})}\frac{\partial(c)}{\partial({}_L\mathbf{V})}\frac{\partial({}_L\mathbf{V})}{\partial(\delta\boldsymbol{\xi})} \qquad (8)
$$

1. The first term is the image gradient on the pixel coordinate ${}_L\mathbf{u}$.

2. The second item is the derivative towards perspective projection from the 3D world vertex ${}_L\mathbf{V}$, (X, Y, Z) onto the 2D pixel position ${}_L\mathbf{u}$, expanded as (u, v) in the coordinate $\underrightarrow{\mathcal{F}}_L$.
The relationship can be represented as:

$$\begin{bmatrix} Zu \\ Zv \\ Z \end{bmatrix} = \begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}. \tag{9}$$

That is to say,

$$u = f_x \frac{X}{Z} + c_x, \tag{10}$$

$$v = f_y \frac{Y}{Z} + c_y. \tag{11}$$

Then the second term in the Jacobian equation would be:

$$\frac{\partial(c)}{\partial({}_L\mathbf{V})} = \begin{bmatrix} \frac{\partial u}{\partial X} & \frac{\partial u}{\partial Y} & \frac{\partial u}{\partial Z} \\ \frac{\partial v}{\partial X} & \frac{\partial v}{\partial Y} & \frac{\partial v}{\partial Z} \end{bmatrix} = \begin{bmatrix} \frac{f_x}{Z} & 0 & -\frac{f_x X}{Z^2} \\ 0 & \frac{f_y}{Z} & -\frac{f_y Y}{Z^2} \end{bmatrix} \tag{12}$$

3. The term is the derivative towards the vertex ${}_L\mathbf{V}$ in the live coordinate $\underrightarrow{\mathcal{F}}_L$ transformed form the ${}_W\mathbf{V}$ under the small perturbation $\delta\boldsymbol{\xi}$ on the transformation $\boldsymbol{T}_{WL}$.

$$\frac{\partial({}_L\mathbf{V})}{\partial(\delta\boldsymbol{\xi})} = \frac{\partial(\boldsymbol{T}_{WL}^{-1}\,{}_W\mathbf{V})}{\partial(\delta\boldsymbol{\xi})} \tag{13}$$

$$= \lim_{\delta\boldsymbol{\xi}\to 0} \frac{\boldsymbol{T}_{WL}^{-1}\,\exp((-\delta\boldsymbol{\xi})^{\wedge})_W\mathbf{V} - \boldsymbol{T}_{WL}^{-1}\,{}_W\mathbf{V}}{\delta\boldsymbol{\xi}} \tag{14}$$

$$\approx \lim_{\delta\boldsymbol{\xi}\to 0} \frac{\boldsymbol{T}_{WL}^{-1}\,(1 - (\delta\boldsymbol{\xi})^{\wedge})_W\mathbf{V} - \boldsymbol{T}_{WL}^{-1}\,{}_W\mathbf{V}}{\delta\boldsymbol{\xi}} \tag{15}$$

$$= \lim_{\delta\boldsymbol{\xi}\to 0} \frac{-\boldsymbol{T}_{WL}^{-1}\,(\delta\boldsymbol{\xi})^{\wedge}\,{}_W\mathbf{V}}{\delta\boldsymbol{\xi}} \tag{16}$$

$$= \lim_{\delta\boldsymbol{\xi}\to 0} \frac{-\begin{bmatrix} \mathbf{C}_{WL}^T & -\mathbf{C}_{WL}^T\,{}_W\mathbf{r}_{WL} \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{\phi}^{\wedge} & \delta\boldsymbol{\rho} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} {}_W\mathbf{V} \\ 1 \end{bmatrix}}{\delta\boldsymbol{\xi}} \tag{17}$$

$$= \lim_{\delta\boldsymbol{\xi}\to 0} \frac{-\begin{bmatrix} \mathbf{C}_{WL}^T & -\mathbf{C}_{WL}^T\,{}_W\mathbf{r}_{WL} \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{\phi}^{\wedge}\,{}_W\mathbf{V} + \delta\boldsymbol{\rho} \\ 0 \end{bmatrix}}{\delta\boldsymbol{\xi}} \tag{18}$$

$$= \lim_{\delta\boldsymbol{\xi}\to 0} \frac{-\begin{bmatrix} \mathbf{C}_{WL}^T\,\delta\boldsymbol{\phi}^{\wedge}\,{}_W\mathbf{V} + \mathbf{C}_{WL}^T\,\delta\boldsymbol{\rho} \\ 0 \end{bmatrix}}{\delta\boldsymbol{\xi}} \tag{19}$$

The approximation in the Eq. 15 is using the property that under the small perturbation, $\exp((-\delta\boldsymbol{\xi})^{\wedge}) \approx 1 - (\delta\boldsymbol{\xi})^{\wedge}$. The notation of ${}_W\mathbf{V}$ is abused here. It is expressed in the homogeneous coordinate from Eq. 13 to 16 and inhomogeneous coordinate from Eq. 17 to 19.

So, for the small perturbation on the translational part $\delta\boldsymbol{\rho}$, the Jacobian would be:

$$\frac{\partial({}_L\mathbf{V})}{\partial(\delta\boldsymbol{\rho})} = \lim_{\delta\boldsymbol{\rho}\to 0} \frac{-\begin{bmatrix} \mathbf{C}_{WL}^T\,\delta\boldsymbol{\phi}^{\wedge}\,{}_W\mathbf{V} + \mathbf{C}_{WL}^T\,\delta\boldsymbol{\rho} \\ 0 \end{bmatrix}}{\delta\boldsymbol{\rho}} \tag{20}$$

$$= \begin{bmatrix} -\mathbf{C}_{WL}^T \\ 0^T \end{bmatrix}. \tag{21}$$

For the small perturbation on the translational part $\delta\boldsymbol{\phi}$, the Jacobian would be:

$$\frac{\partial({}_L\mathbf{V})}{\partial(\delta\boldsymbol{\phi})} = \lim_{\delta\boldsymbol{\phi}\to 0} \frac{-\begin{bmatrix} \mathbf{C}_{WL}^T\,\delta\boldsymbol{\phi}^\wedge\,{}_W\mathbf{V} + \mathbf{C}_{WL}^T\,\delta\boldsymbol{\rho} \\ 0 \end{bmatrix}}{\delta\boldsymbol{\phi}} \tag{22}$$

$$= \lim_{\delta\boldsymbol{\xi}\to 0} \frac{\begin{bmatrix} \mathbf{C}_{WL}^T\,{}_W\mathbf{V}^\wedge\delta\boldsymbol{\phi} \\ 0 \end{bmatrix}}{\delta\boldsymbol{\phi}} \tag{23}$$

$$= \begin{bmatrix} \mathbf{C}_{WL}^T\,{}_W\mathbf{V}^\wedge \\ 0^T \end{bmatrix}. \tag{24}$$

Inserting Eqs. 12, 21, and 24 into Eq. 8, we can get the Jacobian used in this photometric tracking:

$$J = \bigtriangledown_R \mathbf{I}_L({}_L\mathbf{u}) \begin{bmatrix} \frac{f_x}{Z} & 0 & -\frac{f_x X}{Z^2} \\ 0 & \frac{f_y}{Z} & -\frac{f_y Y}{Z^2} \end{bmatrix} \begin{bmatrix} -\mathbf{C}_{WL}^T & \mathbf{C}_{WL}^T\,{}_W\mathbf{V}^\wedge \end{bmatrix} \tag{25}$$

4 RGB-tracker in SO3 expression

$${}_L\mathbf{V}(\mathbf{0}) = \mathbf{C}_{WL}^T\,({}_W\mathbf{V} - {}_W\mathbf{r}_{WL}) \tag{26}$$

With the small perturbation $(\delta\boldsymbol{\rho}, \delta\boldsymbol{\phi})$ performed on the $({}_W\mathbf{r}_{WL}, \mathbf{C}_{WL})$ respectively, the position of ${}_L\mathbf{V}$ would become:

$${}_L\mathbf{V}(\delta\boldsymbol{\xi}) = \left((\mathbf{1} + \delta\boldsymbol{\phi}^\wedge)\mathbf{C}_{WL}\right)^T ({}_W\mathbf{V} - ({}_W\mathbf{r}_{WL} + \delta\boldsymbol{\rho})) \tag{27}$$

The rotation part can be identically expressed as:

$$\left((\mathbf{1} + \delta\boldsymbol{\phi}^\wedge)\mathbf{C}_{WL}\right)^T = \mathbf{C}_{WL}^T\,(\mathbf{1} + \delta\boldsymbol{\phi}^\wedge)^T \tag{28}$$

$$= \mathbf{C}_{WL}^T\,(\mathbf{1} + (\delta\boldsymbol{\phi}^\wedge)^T) \tag{29}$$

$$= \mathbf{C}_{WL}^T\,(\mathbf{1} - (\delta\boldsymbol{\phi})^\wedge) \tag{30}$$

So the perturbed 3D position would be:

$${}_L\mathbf{V}(\delta\boldsymbol{\xi}) = \mathbf{C}_{WL}^T\,(\mathbf{1} - (\delta\boldsymbol{\phi})^\wedge)\,(({}_W\mathbf{V} - {}_W\mathbf{r}_{WL}) - \delta\boldsymbol{\rho}) \tag{31}$$

$$= \mathbf{C}_{WL}^T\,({}_W\mathbf{V} - {}_W\mathbf{r}_{WL}) - \mathbf{C}_{WL}^T\,(\delta\boldsymbol{\phi})^\wedge({}_W\mathbf{V} - {}_W\mathbf{r}_{WL}) - \mathbf{C}_{WL}^T\,\delta\boldsymbol{\rho} + \mathbf{C}_{WL}^T\,(\delta\boldsymbol{\phi})^\wedge\delta\boldsymbol{\rho} \tag{32}$$

The difference after the small perturbation would be:

$${}_L\mathbf{V}(\delta\boldsymbol{\xi}) - {}_L\mathbf{V}(\mathbf{0}) = -\mathbf{C}_{WL}^T\,(\delta\boldsymbol{\phi})^\wedge({}_W\mathbf{V} - {}_W\mathbf{r}_{WL}) - \mathbf{C}_{WL}^T\,\delta\boldsymbol{\rho} + \mathbf{C}_{WL}^T\,(\delta\boldsymbol{\phi})^\wedge\delta\boldsymbol{\rho} \tag{33}$$

$$\approx -\mathbf{C}_{WL}^T\,(\delta\boldsymbol{\phi})^\wedge({}_W\mathbf{V} - {}_W\mathbf{r}_{WL}) - \mathbf{C}_{WL}^T\,\delta\boldsymbol{\rho} \tag{34}$$

The rotational derivative would be:

$$\frac{\partial({}_L\mathbf{V})}{\partial(\delta\boldsymbol{\phi})} = \lim_{\delta\boldsymbol{\phi}\to 0} \frac{{}_L\mathbf{V}(\delta\boldsymbol{\xi}) - {}_L\mathbf{V}(\mathbf{0})}{\delta\boldsymbol{\phi}} \tag{35}$$

$$= \lim_{\delta\boldsymbol{\phi}\to 0} \frac{-\mathbf{C}_{WL}^T\,(\delta\boldsymbol{\phi})^\wedge({}_W\mathbf{V} - {}_W\mathbf{r}_{WL}) - \mathbf{C}_{WL}^T\,\delta\boldsymbol{\rho}}{\delta\boldsymbol{\phi}} \tag{36}$$

$$= \lim_{\delta\boldsymbol{\phi}\to 0} \frac{\mathbf{C}_{WL}^T\,({}_W\mathbf{V} - {}_W\mathbf{r}_{WL})^\wedge(\delta\boldsymbol{\phi})}{\delta\boldsymbol{\phi}} \tag{37}$$

$$= \mathbf{C}_{WL}^T\,({}_W\mathbf{V} - {}_W\mathbf{r}_{WL})^\wedge \tag{38}$$

The translational derivative would be:

$$\frac{\partial({}_L\mathbf{V})}{\partial(\delta\boldsymbol{\rho})} = \lim_{\delta\boldsymbol{\rho}\to 0} \frac{{}_L\mathbf{V}(\delta\boldsymbol{\xi}) - {}_L\mathbf{V}(\mathbf{0})}{\delta\boldsymbol{\rho}} \tag{39}$$

$$= \lim_{\delta\boldsymbol{\rho}\to 0} \frac{-\mathbf{C}_{WL}^T\,(\delta\boldsymbol{\phi})^\wedge({}_W\mathbf{V} - {}_W\mathbf{r}_{WL}) - \mathbf{C}_{WL}^T\,\delta\boldsymbol{\rho}}{\delta\boldsymbol{\rho}} \tag{40}$$

$$= -\mathbf{C}_{WL}^T \tag{41}$$