Guarantees of convergence to a dynamic user equilibrium for a single arc network

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Abstract: While steady state traffic equilibrium problems have been successfully studied, the analysis complicates when the dynamics of the vehicles is taken into account. The literature on the topic presents a variety of models, but usually the algorithms suggested do not possess convergence guarantees. We propose a simple game where all the vehicles travel along the same arc and choose the starting time of their trip. Based on the first Wardrop principle, we formulate the traffic user equilibrium as a variational inequality. Since monotonicity of the corresponding operator guarantees convergence of gradient-based algorithms, we provide theoretical guarantees for short time horizons, and analyze it numerically for longer ones. We conclude with simulations showing that convergence can be achieved also in more general setups, for example with multiple origins or destinations.

Keywords: Dynamic user equilibrium, Convergence guarantees, Traffic road network, Cell transmission model, Monotonicity.

1. INTRODUCTION

Traffic congestion is a well-recognized issue in modern society, and the corresponding economic costs are significant, see Arnott and Small (1994). Traffic engineers have studied the problem for decades from different perspectives, including theoretical and numerical analysis or field experiments (Geroliminis and Daganzo (2008)). Since every driver seeks his own interest (e.g., minimizing the travel time) and is affected by the others choices via congestion, a classic approach is to model the traffic problem as a game (Patriksson (2015)). Different equilibrium models have been presented in order to predict or control congestion. The steady state analysis has been initiated in Wardop (1952) and it has been broadly investigated (Correa and Stier-Moses (2011)), as well as extensively applied to real world applications as in Sheffi (1985). Static models fail to capture important features that are inherently time dependent, such as queue formation or departure time predictions. A formulation of the dynamic user equilibrium problem can be found in Smith (1979) and Friesz et al. (1993) for discrete and continuous time dynamics, respectively. Many works consider a single road network, where the departure time is the only decision variable, see Mahmassani and Herman (1984); Gubins and Verhoef (2012); Fosgerau (2015). Others fix the departure time, but allow players to choose their route, see Smith (1993); Heydecker and Addison (1999).
Despite the variety of models studied, the authors typically focus on existence and uniqueness issues while limiting themselves to propose algorithms that work in practice, but do not possess convergence guarantees as in Huang and Lam (2002); Han et al. (2011). Inspired from the cell-

transmission model of Daganzo (1995) and the dynamic model of Wie et al. (1990), we describe the dynamics of the vehicles on the network as a discrete time positive compartmental system. We assume each user minimizes the sum of his travel time and a discomfort term penalizing earliness or lateness of arrival. We study the dynamic user equilibrium problem arising from such model and allow each user to decide both his departure time and the possible route. Our model does not enforce the well known first-in-first-out (FIFO) condition as in Smith (1993); Ran et al. (1996), since a trade-off has to be made between adherence to reality and mathematical tractability. Our interest is in providing *provable guarantees* for algorithmic convergence to a traffic user equilibrium. We are particularly interested in decentralized algorithms, as these schemes are usually intended for large populations of users. To address the issue, we formulate the dynamic user equilibrium as a variational inequality problem coupled with a non linear constraint representing the dynamic of the system. Our preliminary results are based on monotonicity conditions for the variational inequality operator, obtained in a simplified setup where all users share the same origin and the same destination. We note that the results of this paper can be extended from a single arc network to a parallel arc network, where the users share the same origin and the same destination and they can choose between parallel links. Even if this setup may seem highly idealized, it can describe the daily commuting from a residential area to a central business center, many authors have used similar simplifications when looking at this problem, see Han et al. (2011); Mahmassani and Herman (1984); Gubins and Verhoef (2012); Wie (1993).
The paper is organized as follows. Section 2 formulates the problem, Section 3 shows existence of the dynamic user equilibrium and presents a decentralized algorithm. Sections 4 and 5 provide guarantees for the convergence of the algorithm. Section 6 describes numerical simulations.

2. GAME FORMULATION

2.1 Variables

Let us consider a single-arc network, composed by an origin $\mathcal{O}$, a destination $\mathcal{D}$ and a single arc, which represents a road connecting $\mathcal{O}$ to $\mathcal{D}$. We introduce the set $\mathcal{W} = \{1, \ldots, W\}$ of users. Each user $w \in \mathcal{W}$ intends to send $d^w \in \mathbb{R}_{>0}$ vehicles from $\mathcal{O}$ to $\mathcal{D}$. We consider a time horizon $\mathcal{T} = \{1, \ldots, T\}$ with a discretization interval δt. Each user chooses the non-negative quantities $h_w(t)$ representing the number of vehicles departing at time t. As a consequence of the departures, user w possesses $x^w(t)$ vehicles at time t on the arc. The total number of vehicles on the arc at time t is therefore $\sigma(t) := \sum_{w \in \mathcal{W}} x^w(t)$.

2.2 The fundamental diagram of traffic

A key role in modelling the vehicle dynamics is played by the so-called outflow function $g(\cdot)$, introduced in Daganzo (1995), which maps the total number of vehicles on the arc to the total outflow from the arc. Here the outflow function is based on the *fundamental diagram* of traffic which is presented in Greenshields Symposium (2011) and experimentally justified in Li and Zhang (2011).

Given positive scalars b and c, we define the outflow function $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ as follows:

$$g(\sigma) := \begin{cases} \sigma & \text{if } 0 \leq \sigma < \sigma_{\text{con}} \\ -b\sigma + c & \text{if } \sigma_{\text{con}} \leq \sigma \leq \sigma_{\text{blo}} \\ 0 & \text{if } \sigma > \sigma_{\text{blo}}, \end{cases} \quad (1)$$

where $\sigma_{\text{con}} := \frac{c}{1+b}$ and $\sigma_{\text{blo}} := \frac{c}{b}$, see Figure 1.
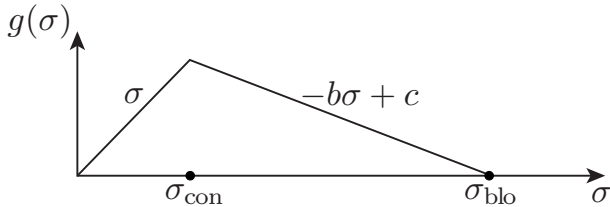

Fig. 1. The outflow function. On the horizontal axis, the total number of vehicles on the arc; on the vertical axis, the total outflow of vehicles from the arc.

When $0 \leq \sigma < \sigma_{\text{con}}$, the vehicles travel in free-flow, i.e. their density is low enough not to slow each other down. The free-flow part of the fundamental diagram has unitary slope, meaning that all the vehicles present on the arc leave it in one time interval δt. This is because we assume, as done in Daganzo (1995), that the arc has a length δd equal to the distance travelled at free-flow speed in one time interval δt.
When $\sigma_{\text{con}} \leq \sigma \leq \sigma_{\text{blo}}$ congestion builds up and the vehicles slow down. When $\sigma > \sigma_{\text{blo}}$, all the vehicles are blocked and cannot proceed.

Field experiments conducted in Li and Zhang (2011) show that b takes values in the range $0.08 \sim 0.19 \, \delta d / \delta t$, while c takes values between $32 \sim 39$ vehicles$/\delta t$ (for a time interval of one minute).

2.3 Vehicle dynamics

The function $g(\cdot)$ determines the total outflow from the arc. In the following we build upon this and derive the dynamics of individual vehicles. To this end, let us assume that the total outflow is split among the users proportionally to their number of vehicles on the arc; consequently the outflow at time t for user w is

$$g(\sigma(t)) \frac{x^w(t)}{\sigma(t)}.$$

To compactly denote $g(\sigma)/\sigma$, we further define $f(\cdot)$ as

$$f(\sigma) := \begin{cases} 1 & \text{if } 0 \leq \sigma < \sigma_{\text{con}} \\ -b + \dfrac{c}{\sigma} & \text{if } \sigma_{\text{con}} \leq \sigma \leq \sigma_{\text{blo}} \\ 0 & \text{if } \sigma > \sigma_{\text{blo}}, \end{cases} \quad (2)$$

which can be used to introduce the users dynamics:

$$x^w(t+1) = x^w(t) - f(\sigma(t))x^w(t) + h^w(t). \quad (3)$$

In words, the number of vehicles of user w on the arc decreases due to the outflow and increases due to new departures. We assume throughout the paper that there are no vehicles on the arc at the beginning of the horizon, i.e. $x^w(0) = 0$ for all w. As a consequence, given the departures for every user w at time t, the dynamics (3) uniquely determine the number of vehicles for all w, t. This correspondence can be captured defining the function $x^w = \chi^w(\mathbf{h})$, which maps the vector $\mathbf{h}$ stacking together $h^w(t)$ for all w, t, to the number of vehicles x^w specific to user w, for all t. In the following we denote with $\boldsymbol{\sigma}$ the vector stacking together $\sigma(t)$ for all t, and we indicate with $\boldsymbol{\sigma}(\mathbf{h})$ the $\boldsymbol{\sigma}$ uniquely determined by $\mathbf{h}$ via the dynamics (3).
Lemma 1. (Dynamics factorization). There exists a matrix $M(\mathbf{h})$ whose entries depend on $\mathbf{h}$ such that for each w

$$\chi^w(\mathbf{h}) = M(\mathbf{h}) \cdot h^w. \quad (4)$$

Proof. Based on the dynamics (3), the expression for $M(\mathbf{h})$ is

$$\begin{bmatrix} x^w(1) \\ x^w(2) \\ x^w(3) \\ \vdots \\ x^w(T) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ (1-f_1) & 1 & 0 & 0 & \ldots & 0 \\ (1-f_1)(1-f_2) & (1-f_2) & 1 & 0 & \ldots & 0 \\ \vdots & & & & \vdots & \\ (1-f_1)\cdots(1-f_{T-1}) & & & & \ldots & 1 \end{bmatrix}}_{M(\mathbf{h})} \begin{bmatrix} h^w(0) \\ h^w(1) \\ h^w(2) \\ \vdots \\ h^w(T-1) \end{bmatrix},$$

$$(5)$$

where we denoted $f_t := f(\sigma(t))$. $\qquad \square$

2.4 Cost function

The goal of each user is to minimize selfishly his own cost, which we model as linear function of x^w:

$$\tilde{c}^w(x^w) := \sum_{t \in \mathcal{T}} \alpha^w(t)x^w(t) := \langle \alpha^w, x^w \rangle, \quad (6)$$

where α^w and x^w are the stacked vectors of cost coefficients and number of vehicles of user w for all t. We assume throughout the paper that $\alpha^w(t) \geq 0$.

This cost, although simple, can describe a range of important objectives, among which:

- *total travel time*: if $\alpha^w = \mathbb{1}_T$ (vector of unit entries), then $\tilde{c}^w(x^w)$ represents the total travel time of the vehicles of user w;

- *arrival time preference*: to approximate the discomfort term penalizing for early or late arrivals with respect to a desired time window, one can select the coefficients with a profile as in the solid line of Figure 2.

Note that in the cost (6) it is possible to account for both the total travel time and the arrival time preference by defining the coefficients α^w to be a weighted average of the coefficients corresponding to the cases just discussed.
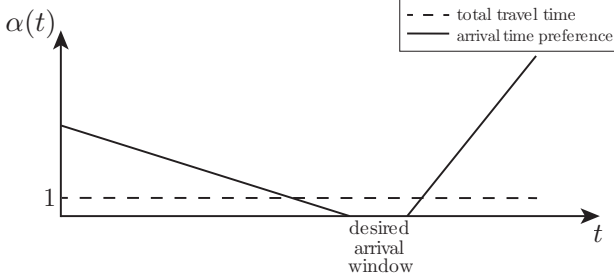

Fig. 2. Two possible choices of the cost coefficients α^w.

By substituting the dynamics (3) into the cost function (6), we obtain the cost as a function of $\mathbf{h}$

$$c^w(\mathbf{h}) \coloneqq \tilde{c}^w(\chi^w(\mathbf{h})) = \langle \alpha^w, \chi^w(\mathbf{h}) \rangle.$$

The factorization (4) of Lemma 1 can be used to define the function $\mathcal{C}^w(\cdot)$:

$$c^w(\mathbf{h}) = \langle \alpha^w, \chi^w(\mathbf{h}) \rangle = \langle \alpha^w, M(\mathbf{h}) \cdot h^w \rangle \coloneqq \langle \mathcal{C}^w(\mathbf{h}), h^w \rangle, \tag{7}$$

where

$$\mathcal{C}^w(\mathbf{h}) \coloneqq M(\mathbf{h})^T \cdot \alpha^w. \tag{8}$$

The cost $c^w(\mathbf{h})$ can thus be expressed as the scalar product between the actions h^w of user w, and the function $\mathcal{C}^w(\mathbf{h})$, which contains all the non-linearity of the dynamics and the interactions among the different users. In the following we refer to $\mathcal{C}^w(\mathbf{h})$ as *cost per action*. Note that, if the dynamics (3) were linear, $\mathcal{C}^w(\mathbf{h})$ would be constant in $\mathbf{h}$.

2.5 Game primitives and user equilibrium

We model the non-cooperative environment described in the previous subsections as a game, which is defined by the following primitives:

- users: $\mathcal{W} = \{1, \dots, W\}$; (9a)
- individual constraint of user w:

$$\mathcal{H}^w \coloneqq \left\{ h^w \middle| h^w(t) \geq 0, \sum_{t \in \mathcal{T}} h^w(t) = d^w \right\}; \tag{9b}$$

- cost of user w: $c^w(\mathbf{h}) = \langle \mathcal{C}^w(\mathbf{h}), h^w \rangle$. (9c)

A key concept in traffic equilibrium is the *dynamic user equilibrium* (DUE), which is defined as in the following.

Definition 1. A set of strategies $\mathbf{h}^\star = [h^{1\star}, \dots, h^{W\star}]$ is a dynamic user equilibrium of the game in (9) if for all users $w \in \mathcal{W}$ it holds $h^{w\star} \in \mathcal{H}^w$ and

$$\langle \mathcal{C}^w(\mathbf{h}^\star), h^{w\star} \rangle \leq \langle \mathcal{C}^w(\mathbf{h}^\star), h^w \rangle, \quad \forall h^w \in \mathcal{H}^w. \qquad \square$$

Intuitively, a set of strategies $\mathbf{h}^\star$ is a DUE if it satisfies the individual constraints and, considering the cost per action as a fixed quantity, no user can improve his cost by unilaterally deviating from his own strategy. One can show that Definition 1 is equivalent to the following more

classic Definition 2, which is based on Ran et al. (1996). The proof can be found in Dafermos (1980) for the static user equilibrium problem, but the extension to our setup is straightforward.

Definition 2. At dynamic user equilibrium, user w sends vehicles at time t, i.e. sets $h^{w\star}(t) > 0$, if the corresponding cost per action in (7) is minimum. As a consequence, the actions used (i.e., those with positive vehicle departures) have equal costs, and the actions with higher costs will not be used. $\qquad \square$

3. EXISTENCE AND DECENTRALIZED ALGORITHM

In order to study existence of a DUE we define the set $\mathcal{H}$ and the operator $\mathcal{C}(\cdot)$ as follows:

$$\mathcal{H} \coloneqq \mathcal{H}^1 \times \cdots \times \mathcal{H}^W, \quad \mathcal{C}(\mathbf{h}) \coloneqq [\mathcal{C}^1(\mathbf{h}), \dots, \mathcal{C}^W(\mathbf{h})]^\top. \tag{10}$$

Proposition 1. The game (9) admits a dynamic user equilibrium.

Proof. Using Definition 1 it is straightforward to show that $\mathbf{h}^\star$ is a DUE if and only if $\mathbf{h}^\star$ is a solution of the variational inequality VI$(\mathcal{H}, \mathcal{C}(\mathbf{h}))$[1]. The existence result follows by (Facchinei and Pang, 2000, Corollary 2.2.5), upon noticing that $\mathcal{H}$ is compact and convex by (9b), and $\mathcal{C}(\mathbf{h})$ is a continuous function of $\mathbf{h}$. $\qquad \square$

In general the DUE is not unique. While a large number of algorithms have been proposed in order to solve the variational inequality problem, their convergence is often based on modifications of the concept of monotone operators. To achieve a DUE, we introduce the extragradient algorithm (Facchinei and Pang, 2000, Algorithm 12.1.9). We point out that this is one of many algorithms of (Facchinei and Pang, 2000, Chapter 12), and any of them would work.

Algorithm 1 Extra-gradient algorithm

Initialization Set $k \leftarrow 0$. Choose $\tau > 0$ (step size) small enough, $\epsilon > 0$ (a tolerance threshold), $h_{[k]} \in \mathcal{H}$ (initial condition).
while $\|h_{[k]} - h_{[k-1]}\| < \epsilon$ **do**
$h^w_{\text{temp}} \leftarrow \texttt{Proj}_{\mathcal{H}^w}(h^w_{[k]} - \tau\,\mathcal{C}^w(\boldsymbol{h}_{[k]}))$ for all w
$h^w_{[k+1]} \leftarrow \texttt{Proj}_{\mathcal{H}^w}(h^w_{[k]} - \tau\,\mathcal{C}^w(\boldsymbol{h}_{\text{temp}}))$ for all w
$k \leftarrow k + 1$

Since $\mathcal{C}^w(\cdot)$ can be expressed as a function of the sole $\boldsymbol{\sigma}$ via (8) and (4), and since $\texttt{Proj}_{\mathcal{H}^w}$ can be decoupled into individual projections, the previous algorithm can be decentralized (parallelized). If a central agent gathers the departures of all the users and broadcasts the resulting $\boldsymbol{\sigma}$, each user can independently update his actions.

To establish conditions for the convergence of Algorithm 1, we introduce the definition of monotone operator.

Definition 3. The operator $\mathcal{C}(\cdot)$ is monotone on $\mathcal{H}$ if and only if

$$\langle \mathcal{C}(\mathbf{h}) - \mathcal{C}(\hat{\mathbf{h}}), \mathbf{h} - \hat{\mathbf{h}} \rangle \geq 0, \quad \forall \mathbf{h}, \hat{\mathbf{h}} \in \mathcal{H}. \qquad \square$$

Proposition 2. Assume that the operator $\mathcal{C}(\cdot)$ is monotone and $\tau < \frac{1}{L_{\mathcal{C}}}$, with $L_{\mathcal{C}}$ being the Lipschitz constant of $\mathcal{C}$.

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[1] For a definition of variational inequality, see (Facchinei and Pang, 2000, Definition 1.1.1).

Then the extra-gradient algorithm converges to a dynamic user equilibrium of the game (9).

The proof can be found in (Facchinei and Pang, 2000, Theorem 12.1.11).

4. REDUCTION TO THE CONGESTED REGION

This section, along with the next one, is devoted to the derivation of conditions on the non-negative cost coefficients α^w in (6) under which the operator $\mathcal{C}(\cdot)$ in (10) is monotone. We assume in Sections 4 and 5 that each user features the same cost coefficients $\alpha^w = \alpha$ in (6). As a consequence, all users w have the same cost per action $\mathcal{C}^w = \mathcal{C}$ (see (7)). The users are still heterogeneous, as they might differ in the demands d^w. We intend to show that the cost per action vector is a sole function of the sum of the vehicle injections. To this end, let us define $s := \sum_{w \in \mathcal{W}} h^w$.

Lemma 2. The operator $\mathcal{C}(\cdot)$ can be expressed as

$$\mathcal{C}(\mathbf{h}) = \mathbb{1}_W \otimes \mathcal{C}(s), \qquad (11)$$

where, for any user w, $\mathcal{C}(\cdot)$ is the cost per action of player w as in (8), which is the same for all w and depends on s.

Proof. Homogeneity of the users' cost coefficients α guarantees that the cost per action does not depend on the specific player w, and so $\mathcal{C}(\cdot)$ can be expressed as the vertical repetition of the same vector $\mathcal{C}$. By (8), $\mathcal{C}(\cdot)$ is a linear combination of the entries of $M(\mathbf{h})$ in (5), hence we are left to show that $M(\mathbf{h})$ depends on s only. Since every entry of $M(\mathbf{h})$ is a function of $\{f_t\}_{t \in \mathcal{T}}$, and $f_t = f(\sigma(t))$, the proof is concluded if we show that any $\sigma(t)$ is a sole function of s. Summing (5) over all users w, and denoting $f_t := f(\sigma(t))$, one obtains

$$\begin{bmatrix} \sigma(1) \\ \sigma(2) \\ \sigma(3) \\ \vdots \\ \sigma(T) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0\ 0\ \ldots\ 0 \\ (1-f_1) & 1 & 0\ 0\ \ldots\ 0 \\ (1-f_1)(1-f_2) & (1-f_2) & 1\ 0\ \ldots\ 0 \\ \vdots & & \\ (1-f_1)\cdots(1-f_{T-1}) & & \ldots\ 1 \end{bmatrix} \begin{bmatrix} s(0) \\ s(1) \\ s(2) \\ \vdots \\ s(T{-}1) \end{bmatrix}, \qquad (12)$$

expressing the fact that $\sigma(t+1)$ depends only on $\sigma(t)$ and $s(t)$. By induction one proves that $\sigma(t)$ is a function of s only, as required. $\qquad\square$

Since we are interested in guaranteeing monotonicity of $\mathcal{C}(\cdot)$ under conditions that are independent from the demands $\{d^w\}_{w \in \mathcal{W}}$, in the following we study montonicity of $\mathcal{C}(\cdot)$ in $\mathbb{R}_{\geq 0}^{W \cdot T \cdot A}$, which ensures monotonicity in $\mathcal{H}$.

Lemma 3. The operator $\mathcal{C}(\cdot)$ is monotone in $\mathbb{R}_{\geq 0}^{W \cdot T}$ if and only if the operator $\mathcal{C}(\cdot)$, is monotone in $\mathbb{R}_{\geq 0}^T$.

Proof. The decomposition of $\mathcal{C}(\cdot)$ in (11) yields

$$\langle \mathcal{C}(\mathbf{h}) - \mathcal{C}(\hat{\mathbf{h}}), \mathbf{h} - \hat{\mathbf{h}} \rangle = \langle \mathbb{1}_W \otimes (\mathcal{C}(s) - \mathcal{C}(\hat{s})), \mathbf{h} - \hat{\mathbf{h}} \rangle =$$

$$\langle \mathcal{C}(s) - \mathcal{C}(\hat{s}), \sum_{w=1}^W (h^w - \hat{h}^w) \rangle = \langle \mathcal{C}(s) - \mathcal{C}(\hat{s}), s - \hat{s} \rangle.$$

Using definition 3, one concludes the proof. $\qquad\square$

We proceed to study monotonicity of $\mathcal{C}(\cdot)$ in $\mathbb{R}_{\geq 0}^T$. We introduce the subscript T as in $\mathcal{C}_T(\cdot)$ to highlight the length T of the time horizon.

To study monotonicity of $\mathcal{C}_T(\cdot)$, we note that at any time instant the arc can be either in free flow ($0 \leq \sigma \leq \sigma_{\mathrm{con}}$),

congested ($\sigma_{\mathrm{con}} \leq \sigma < \sigma_{\mathrm{blo}}$) or blocked ($\sigma \geq \sigma_{\mathrm{blo}}$), hence we express $\mathbb{R}_{\geq 0}^T$ as the union of 3^T closed regions: $\mathbb{R}_{\geq 0}^T = \bigcup_{r=1}^{3^T} \mathcal{R}_r$. Based on this, the next lemma gives a necessary and sufficient condition for monotonicity of $\mathcal{C}_T(\cdot)$.

Lemma 4. The operator $\mathcal{C}_T(\cdot)$ is monotone in $\mathbb{R}_{\geq 0}^T$ if and only if it is monotone in every region $\mathcal{R}_r$.

Proof. According to the definition of subgradient in Schaible et al. (1996), monotonicity in $\mathbb{R}_{\geq 0}^T$ is equivalent to positive semi-definiteness of the subgradients of $\mathcal{C}_T(\cdot)$ in every point $s \in \mathbb{R}_{\geq 0}^T$. This, in turn, is equivalent to having the subgradients of $\mathcal{C}_T(\cdot)$ positive-semidefinite in the interior of every region $\mathcal{R}_r$, which is equivalent to monotonicity of $\mathcal{C}_T(\cdot)$ in $\mathcal{R}_r$. $\qquad\square$

As in the proof of Lemma 2, we denote $f_t := f(\sigma(t))$. According to (8), the operator $\mathcal{C}_T(s)$ can be expressed as $M(\mathbf{h})^\top \alpha$, where $M(\mathbf{h})$ is defined in (5). This yields

$$\mathcal{C}_T(s) = \begin{bmatrix} 1 & (1{-}f_1) & (1{-}f_1)(1{-}f_2) & \ldots & (1{-}f_1)\cdots(1{-}f_{T-1}) \\ 0 & 1 & (1-f_2) & \ldots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_T \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 + \alpha_2(1-f_1) + \alpha_3(1-f_1)(1-f_2) + \ldots + \\ \qquad \alpha_T(1-f_1)\ldots(1-f_{T-1}) \\ \alpha_2 + \alpha_3(1-f_2) + \ldots + \alpha_T(1-f_2)\ldots(1-f_{T-1}) \\ \alpha_3 + \alpha_4(1-f_3) + \ldots + \alpha_T(1-f_3)\ldots(1-f_{T-1}) \\ \vdots \\ \alpha_{T-1} + \alpha_T(1-f_{T-1}) \\ \alpha_T \end{bmatrix} \quad (13)$$

To state the next theorem, let us introduce the sets

$$\mathcal{R}_{\mathrm{CON}} := \{ s \in \mathbb{R}_{\geq 0}^T | \sigma_{\mathrm{con}} \leq \sigma(t) \leq \sigma_{\mathrm{blo}}, \forall t \in \mathcal{T} \} \quad (14a)$$

$$\mathcal{A}_T := \{ \alpha \in \mathbb{R}_{\geq 0}^T | \mathcal{C}_T(s) \text{ is MON}, \forall s \in \mathbb{R}_{\geq 0}^T \} \quad (14b)$$

$$\mathcal{A}_T^{\mathrm{con}} := \{ \alpha \in \mathbb{R}_{\geq 0}^T | \mathcal{C}_T(s) \text{ is MON}, \forall s \in \mathcal{R}_{\mathrm{CON}} \}. (14c)$$

The following theorem gives conditions on the cost vector α_T which guarantee monotonicity of $\mathcal{C}_T(s)$.

Theorem 3. The coefficient vector $\alpha_T = [\alpha_1, \ldots, \alpha_T] \in \mathcal{A}_T$ if and only if the following two conditions hold for all t_1, t_2 such that $0 \leq t_1 \leq t_2 \leq T$:

i) $[\alpha_{t_1}, \ldots, \alpha_{t_2}] \in \mathcal{A}_{t_2 - t_1 + 1}^{\mathrm{con}}$;

ii) $[\alpha_{t_1}, \ldots, \alpha_{t_2 - 1}, \alpha_{t_2} + \alpha_{t_2 + 1} + \cdots + \alpha_T] \in \mathcal{A}_{t_2 - t_1 + 1}^{\mathrm{con}}$.

Proof. We only prove that $\alpha \in \mathcal{A}_T \Rightarrow$ i) and ii) as the other implication requires a more cumbersome notation.

"$\alpha \in \mathcal{A}_T \Rightarrow$ i)": Consider $\sigma(t) \in \mathrm{CON} := \{ \sigma \in \mathbb{R}_{\geq 0} | \sigma_{\mathrm{con}} \leq \sigma \leq \sigma_{\mathrm{blo}} \}$ for all t, except for $\tilde{t}$, where $\sigma(\tilde{t}) \in \mathrm{FREE} := \{ \sigma \in \mathbb{R}_{\geq 0} | 0 \leq \sigma < \sigma_{\mathrm{con}} \}$. Since $f_{\tilde{t}} = 1$, $\mathcal{C}_T(s)$ in (13) becomes

$$\mathcal{C}_T(s) = \begin{bmatrix} \alpha_1 + \alpha_2(1-f_1) + \alpha_3(1-f_1)(1-f_2) + \ldots + \\ \qquad \alpha_{\tilde{t}}(1-f_1)\ldots(1-f_{\tilde{t}-1}) \\ \alpha_2 + \alpha_3(1-f_2) + \ldots + \alpha_{\tilde{t}}(1-f_2)\ldots(1-f_{\tilde{t}-1}) \\ \vdots \\ \alpha_{\tilde{t}-1} + \alpha_{\tilde{t}}(1-f_{\tilde{t}-1}) \\ \alpha_{\tilde{t}} \\ \alpha_{\tilde{t}+1} + \alpha_{\tilde{t}+2}(1-f_{\tilde{t}+1}) + \ldots + \\ \qquad \alpha_T(1-f_{\tilde{t}+1})\ldots(1-f_T) \\ \vdots \\ \alpha_T \end{bmatrix}.$$

The vector $\mathcal{C}_T(s)$ can be split into two subvectors: one obtained with the first $\tilde{t}$ components of $\mathcal{C}_T(s)$, and the other with the last $T - \tilde{t}$ components of $\mathcal{C}_T(s)$. The first subvector depends on $[\sigma_1, \ldots, \sigma_{\tilde{t}-1}]$ and thus only on $[s_0, \ldots, s_{\tilde{t}-2}]$, due to (12). The fact that $f_{\tilde{t}} = 1$ makes $\sigma_{\tilde{t}+1}$ independent from $s_{\tilde{t}-1}$ (see (12)), hence the second subvector depends only on $[s_{\tilde{t}}, \ldots, s_{T-1}]$. From the definition of monotonicity of $\mathcal{C}_T(\cdot)$, it follows that

- the first subvector must be MON in $[s_0, \ldots, s_{\tilde{t}-1}]$, hence it is necessary $[\alpha_1, \ldots, \alpha_{\tilde{t}}] \in \mathcal{A}_{\tilde{t}}^{\mathrm{con}}$;
- the second subvector must be MON in $[s_{\tilde{t}}, \ldots, s_T]$, hence it is necessary $[\alpha_{\tilde{t}+1}, \ldots, \alpha_T] \in \mathcal{A}_{T-\tilde{t}}^{\mathrm{con}}$.

This proves "$\alpha \in \mathcal{A}_T \Rightarrow$ i)" for $t_1 = 1, t_2 = \tilde{t}$, and for $t_1 = \tilde{t} + 1, t_2 = T$. Generalizing to arbitrary t_1, t_2 is easy.

"$\alpha \in \mathcal{A}_T \Rightarrow$ ii)": Consider $\sigma(t) \in \mathrm{CON}$ for $t < \tilde{t}$ and $\sigma(\tilde{t}) \in \mathrm{BLO} := \{\sigma \in \mathbb{R}_{\geq 0} | \sigma_{\mathrm{blo}} < \sigma\}$. Since $f_{\tilde{t}} = 0$, one can show that $f_{\tilde{t}+1} = \cdots = f_T = 0$. Hence, $\mathcal{C}_T(s)$ becomes

$$\mathcal{C}_T(s) = \begin{bmatrix} \alpha_1 + \alpha_2(1 - f_1) + \alpha_3(1 - f_1)(1 - f_2) + \ldots + \\ \left(\alpha_{\tilde{t}} + \alpha_{\tilde{t}+1} + \ldots + \alpha_T\right)(1 - f_1)\ldots(1 - f_{\tilde{t}-1}) \\ \alpha_2 + \alpha_3(1 - f_2) + \ldots + \\ \left(\alpha_{\tilde{t}} + \alpha_{\tilde{t}+1} + \ldots + \alpha_T\right)(1 - f_2)\ldots(1 - f_{\tilde{t}-1}) \\ \vdots \\ \alpha_{\tilde{t}-1} + \left(\alpha_{\tilde{t}} + \alpha_{\tilde{t}+1} + \ldots + \alpha_T\right)(1 - f_{\tilde{t}-1}) \\ \alpha_{\tilde{t}} + \alpha_{\tilde{t}+1} + \ldots + \alpha_T \\ \alpha_{\tilde{t}+1} + \alpha_{\tilde{t}+2} + \ldots + \alpha_T \\ \vdots \\ \alpha_T \end{bmatrix}.$$

The vector $\mathcal{C}_T(s)$ can thus be split into two subvectors: one obtained with the first $\tilde{t}$ components (that only depend on $[s_0, \ldots, s_{\tilde{t}-2}]$) of $\mathcal{C}_T(s)$, and the other with the last $T - \tilde{t} + 1$ components of $\mathcal{C}_T(s)$ (that are constant in s). From the definition of monotonicity of $\mathcal{C}_T(\cdot)$, it follows that the first subvector must be monotone in $[s_0, \ldots, s_{\tilde{t}-1}]$, hence it is necessary that $[\alpha_1, \ldots, \alpha_{\tilde{t}-1}, \alpha_{\tilde{t}} + \alpha_{\tilde{t}+1} + \ldots + \alpha_T] \in \mathcal{A}_{\tilde{t}-1}^{\mathrm{con}}$. This proves "$\alpha \in \mathcal{A}_T \Rightarrow$ ii)" for $t_1 = 1, t_2 = \tilde{t}$. It is straightforward to generalize to any arbitrary t_1, t_2. $\qquad\square$

5. MONOTONICITY FOR HORIZONS OF LENGTH UP TO 3

We provide conditions on the cost coefficients α that guarantee monotonicity of $\mathcal{C}_T(\cdot)$ in the congested region $\mathcal{R}_{\mathrm{CON}}$. Conditions for monotonicity in $\mathbb{R}_{\geq 0}^T$ can be then derived by using Theorem 3. The operator $\mathcal{C}_T(\cdot)$ is continuously differentiable in $\mathcal{R}_{\mathrm{CON}}$, hence its monotonicity is equivalent to the positive semi-definiteness of its Jacobian $\nabla_s \mathcal{C}_T(s)$, for all $s \in \mathcal{R}_{\mathrm{CON}}$, as proved in (Schaible et al., 1996, Proposition 2.1). For all $\sigma_{\mathrm{con}} \leq \sigma(t) \leq \sigma_{\mathrm{blo}}$, we denote $f_t' := \frac{\partial f(\sigma)}{\partial \sigma}|_{\sigma = \sigma(t)}$.

5.1 Horizon of length $T = 1$ and horizon of length $T = 2$

In the case $T = 1$, expression (13) becomes $\mathcal{C}_T(s) = \alpha_1$. Since $\mathcal{C}_T(s)$ is constant, it is clearly monotone for all α_1.

In the case $T = 2$, expression (13) becomes $\mathcal{C}_T(s) = [\alpha_1 + \alpha_2(1 - f_1), \ \alpha_2]^T$ and therefore

$$\nabla_s \mathcal{C}_T(s) = \begin{bmatrix} \alpha_2(-f_1') & 0 \\ 0 & 0 \end{bmatrix} \succeq 0,$$

since, by definition (2), $f_t'(\sigma) \leq 0$, for all $\sigma \geq 0$ (and in particular for $\sigma \in \mathrm{CON}$) and for all t (and in particular for $t = 1$). Consequently $\mathcal{C}_T(\cdot)$ is monotone independently from $[\alpha_1, \alpha_2]$.

5.2 Horizon of length $T = 3$

In the case $T = 3$, expression (13) becomes

$$\mathcal{C}_T(s) = \begin{bmatrix} \alpha_1 + \alpha_2(1 - f_1) + \alpha_3(1 - f_1)(1 - f_2) \\ \alpha_2 + \alpha_3(1 - f_2) \\ \alpha_3 \end{bmatrix}.$$

Simple algebraic computations lead to

$$\nabla_s \mathcal{C}_T(s) = \begin{bmatrix} \begin{matrix}(-f_1')(\alpha_2 + \alpha_3(1 - f_2)) + \\ \alpha_3(-f_2')(-f_1')\sigma_1(1 - f_1)\end{matrix} & \alpha_3(-f_2')(1 - f_1) & 0 \\ \alpha_3(-f_2')(-f_1')\sigma_1 & \alpha_3(-f_2') & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is enough to study the 2×2 top-left matrix. Note that

- the trace is non-negative since $f_t'(\sigma) \leq 0, \forall \sigma \geq 0, \forall t$;
- the determinant is non-negative if and only if

$$4(\alpha_2 + \alpha_3(1 - f_2)) \geq \alpha_3 f_2' f_1' \sigma_1^2 \qquad (15)$$

holds for all $\sigma_1, \sigma_2 \in \mathrm{CON}$; algebraic operations show

(15) holds for $\sigma_1 = \sigma_2 = \sigma_{\mathrm{CON}} \Rightarrow$

(15) holds for all $\sigma_1, \sigma_2 \in \mathrm{CON}$,

where $\sigma_{\mathrm{CON}} = \frac{c}{1+b}$, according to the outflow (1).

It follows that, to guarantee $\nabla_s \mathcal{C}_T(s)$ for all $s \in \mathcal{R}_{\mathrm{CON}}$, it suffices to check condition (15) for only one point, namely $[\sigma_1, \sigma_2] = [\sigma_{\mathrm{CON}}, \sigma_{\mathrm{CON}}]$. Inserting $\sigma_{\mathrm{CON}} = \frac{c}{1+b}$ into condition (15) results in the following proposition.

Proposition 4. For the case $T = 3$, $\mathcal{C}_T(\cdot)$ is monotone if and only if the following condition on $[\alpha_1, \alpha_2, \alpha_3]$ holds:

$$\frac{\alpha_2}{\alpha_3} \geq \frac{(1 + b)^2}{4}, \qquad\qquad\square$$

6. SIMULATIONS

The game in (9) can be extended to consider a more general network topology. Despite the lack of convergence guarantees, we consider the network of Figure 3 and 3 users willing to travel from $\mathcal{O}^w$ to $\mathcal{D}^w$, each with a demand of $d = 250$. The dynamics are the natural extension of (3) to the network setup. The time horizon is $T = 55$, but users are not allowed to send vehicles after $t = 40$ in order to empty the network before the end. Each road features the same outflow parameters $b = 0.2$ and $c = 40$. Each user's cost takes into account both time spent on the network and the discomfort for early or late arrival, with the coefficients of the arrival time preference in Figure 2 imposed only on the last arc before $\mathcal{D}^w$. The desired arrival windows are respectively $[18, 22]$, $[21, 25]$ and $[16, 20]$. A stopping criterion of $\sum_w \langle \mathcal{C}^w(\mathbf{h}) - \min_p [\mathcal{C}^w(\mathbf{h})] \mathbb{1}, h^w \rangle \leq 10^{-6} \|\mathbf{h}\| \|\mathbf{C}(\mathbf{h})\|$ was set. We used this criterion since, by Definition 2, the left-hand side tends to zero if and only if we approach a dynamic user equilibrium. The algorithm ran 6657 iterations with a step size of $\tau = 0.5$. The time evolution of $\sigma(t)$ on the links is represented in Figure 3, while the departures and the cost per actions of user 3 are depicted in Figure 4. Figure 4 indicates that the vehicles injection of user 3 occurs when the cost per action is minimal, in accordance with the definition of DUE. Note that he manages to get at the destination before the end of his arrival window $t = 20$, which is not the case for the other users, as per Figure 3. To achieve this, user 3 leaves earlier than he would have without congestion.
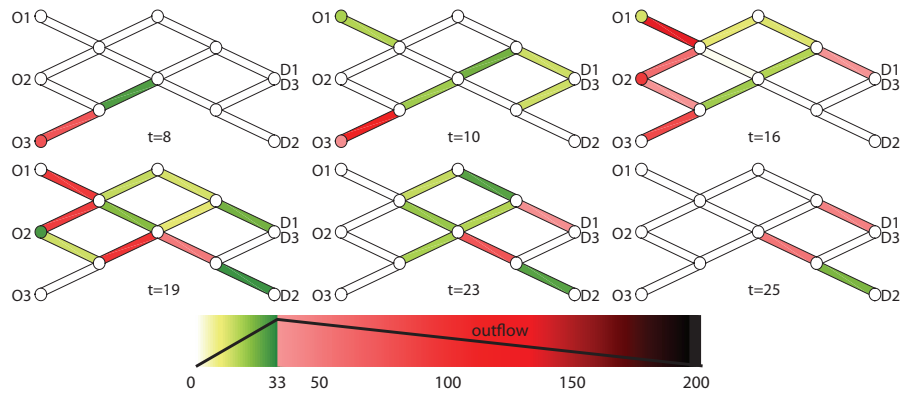
Fig. 3. Evolution of vehicle densities over time. Green tones represent free flow condition, while red indicates congestion.
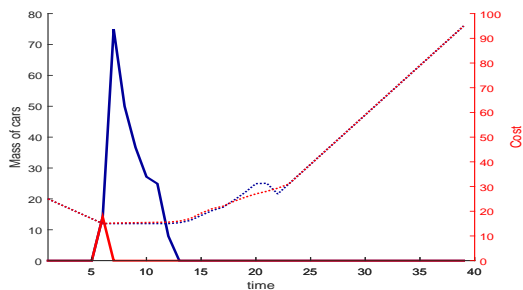

Fig. 4. Cost per action (dashed) and vehicle injections (solid) for user 3 on his two feasible paths (red and blue), as functions of time.

7. CONCLUSIONS

Future work includes theoretical and numerical study of more general networks and longer time horizons.

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