Nash and Wardrop equilibria: convergence and efficiency

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In collaboration with: F. Parise[2], J. Lygeros[1]

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Outline

- Aggregative games
- Convergence between Nash and Wardrop
- Efficiency of equilibria

Motivation

Analysis and **control** of **large scale competitive systems**

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$\hat{x}$ Nash equilibrium

$$J^i(\hat{x}^i, \sigma(\hat{x})) \leq J^i(x^i, \sigma(x^i, \hat{x}^{-i}))$$

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easier to study

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What is the relation between $\hat{x}$ and $\bar{x}$?

Related works

- Wardrop eq. coincides with deterministic mean field/ aggregative eq.

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- distance "between strategies" at Nash $\hat{x}$ and Wardrop $\bar{x}$

A. Haurie and P. Marcotte. "On the relationship between Nash-Cournot and Wardrop equilibria". *Networks*, 1985.

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Nash operator

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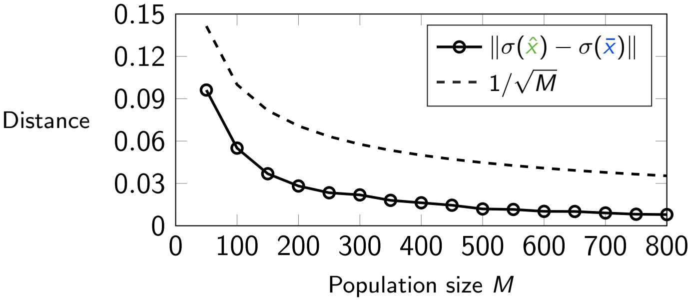

Proof sketch

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Step 1: $\hat{x}$ is a Nash equilibrium $\iff \hat{F}(\hat{x})^\top (x - \hat{x}) \geq 0 \ \ \forall\, x \in \mathcal{X}$

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Step 3: When operators are close, solutions are close

$$||\hat{x} - \bar{x}|| \leq \text{const}''||\hat{F}(\bar{x}) - \bar{F}(\bar{x})||$$

Consequences of $||\hat{x} - \bar{x}|| \leq \frac{\mathrm{const}}{\sqrt{M}}$

equilibrium computation

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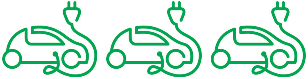
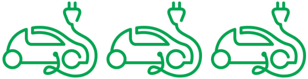


- A fleet of EVs to recharge

Equilibrium efficiency: electric vehicle charging



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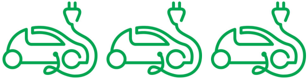

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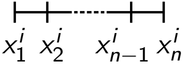

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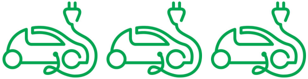

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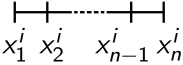

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cost of i: $p(\sigma(x) + d)^\top x^i$

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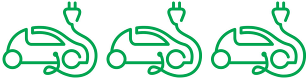

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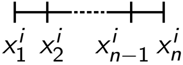

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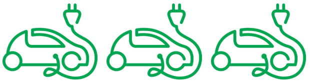

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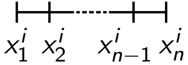

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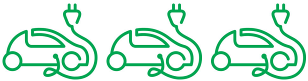

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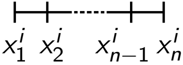

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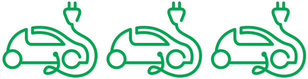

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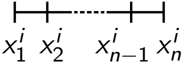

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$$\min_{x \in \mathcal{X}} J_s(x) = p(\sigma(x)+d)^\top(\sigma(x)+d)$$

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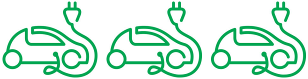

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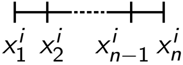

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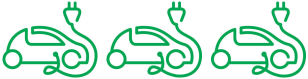
How much does selfish behaviour degrade the performance?

Equilibrium efficiency: electric vehicle charging



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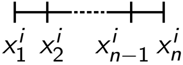

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How much does selfish behaviour degrade the performance?

$$\mathrm{PoA} = \frac{\max_{x \in \mathrm{NE(G)}} J_s(x)}{\min_{x \in \mathcal{X}} J_s(x)} \geq 1$$

Related works

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L. Deori, K. Margellos and M. Prandini. “Price of anarchy in electric vehicle charging control games: When Nash equilibria achieve social welfare”. *Automatica*, 2018.

A. De Paola, D. Angeli and G. Strbac. “Convergence and optimality of a new iterative price-based scheme for distributed coordination of flexible loads in the electricity market”. *IEEE Conference on Decision and Control*, 2017.

M. Gonzales, S. Grammatico and J. Lygeros. “On the price of being selfish in large populations of plug-in electric vehicles”. *IEEE Conference on Decision and Control*, 2015.

O. Beaude, S. Lasaulce and M. Hennebel. “Charging games in networks of electrical vehicles”. *NetGCooP*, 2012.

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Theorem (Equilibrium efficiency)

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Assume regularity + price at time t depends on consumption at time t

$$p(z + d) = [g(z_1 + d_1); \ldots ; g(z_n + d_n)], \qquad g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$$

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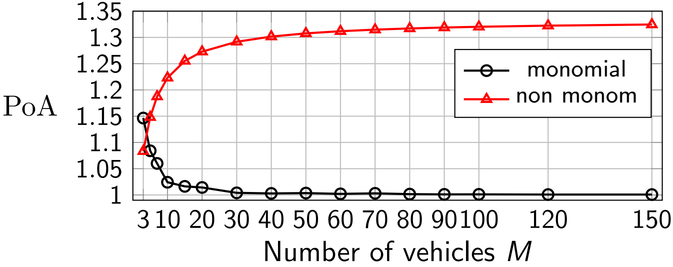
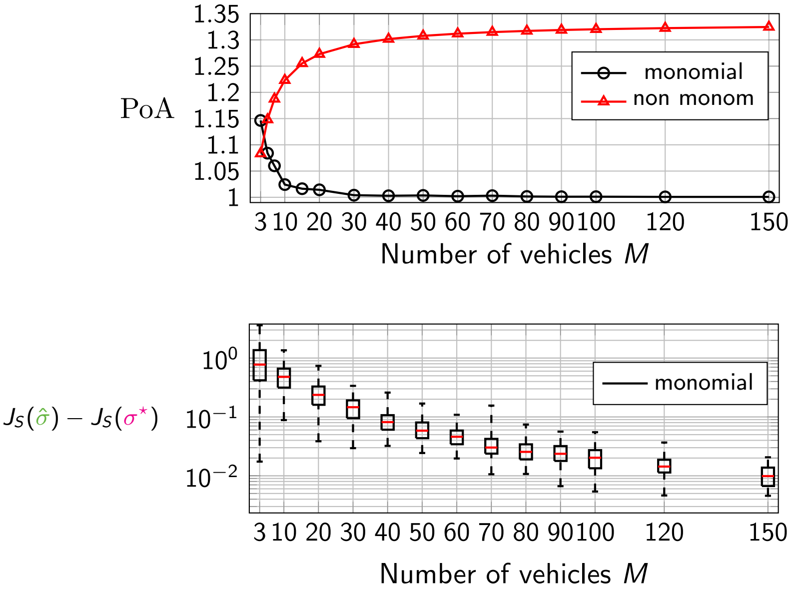
[L-CSS18] includes $p(z+d) = [g_1(z_1+d_1); \dots; g_n(z_n+d_n)]$ time dep.
includes $p(z+d) = C(z+d)$ linear

Numerics validate the result



Numerics validate the result



Proof sketch

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Step 1: $\bar{x}$ is a Wardrop equilibrium $\iff \bar{F}(\bar{x})^\top(x-\bar{x}) \geq 0 \;\; \forall x \in \mathcal{X}$

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Where

$$\bar{F}(\sigma) = [p(\sigma+d)]_{i=1}^{M}$$

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Step 2: $\bar{x}$ coincides with $x^\star$ (for any instance) iff in every point

$$\bar{F}(\sigma) \parallel F^\star(\sigma) \iff \bar{F}(\sigma) = \beta(\sigma)F^\star(\sigma), \quad \beta(\sigma) > 0$$

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Step 3: previous convergence result $\hat{\sigma} \to \bar{\sigma}$ as $M \to \infty$.

Thus $J_s(\hat{\sigma}) \to J_s(\bar{\sigma})$ as $M \to \infty$

so that Nash equilibria become efficient for large M. □

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- Stochasticity and data

Thank you

[L-CSS18] D. Paccagnan, F. Parise and J. Lygeros. "On the Efficiency of Nash Equilibria in Aggregative Charging Games". *IEEE Control Systems Letters*, **2018**.

[TAC18] D. Paccagnan⋆, B. Gentile⋆, F. Parise⋆, M. Kamgarpour, and J. Lygeros. "Nash and Wardrop equilibria in aggregative games with coupling constraints". *IEEE Transactions on Automatic Control*, **2018**.