UC SANTA BARBARA

UNIVERSITÀ DEGLI STUDI DI BRESCIA

The Scenario Approach Meets Uncertain Game Theory and Variational Inequalities

Dario Paccagnan

In collaboration with M.C. Campi

Theme of this talk: take decision based on data and quantify their risk

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data ⟶ decision making process ⟶ solution

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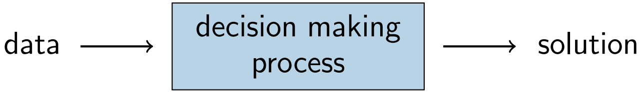

if decision making = optimization problem $\Longrightarrow$

- scenario approach
- robust optimization
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What if decision making is not an optimization problem?

in this talk: **decision making process = variational inequality**

Why variational inequalities?

"*[...] a multitude of interesting connections to numerous disciplines, and a wide range of important applications in engineering and economics*"

F. Facchinei, J-S Pang

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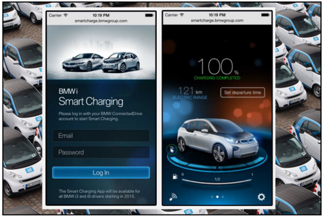
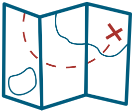
transportation networks

contact problems



demand-response markets

option pricing



ROADMAP

1. Robust variational inequalities + scenario approach

  ⇝ probabilistic bounds on the risk

  ⇝ extension to quasi variational inequalities

2. Uncertain and robust games

  ⇝ how likely that a Nash equilibrium remains such?

  ⇝ application to demand-response

3. Outlook and opportunities

Variational inequalities

Definition (VI): given set $\mathcal{X} \subset \mathbb{R}^n$ and operator $F : \mathcal{X} \to \mathbb{R}^n$,
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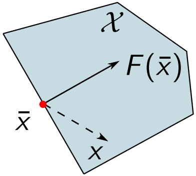

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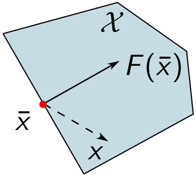

▷ convex optimization as a special case:

$$\bar{x} \text{ solution of } \min_{x \in \mathcal{X}} g(x) \quad \Longleftrightarrow \quad \nabla g(\bar{x})^\top (x - \bar{x}) \geq 0 \;\; \forall x \in \mathcal{X}$$

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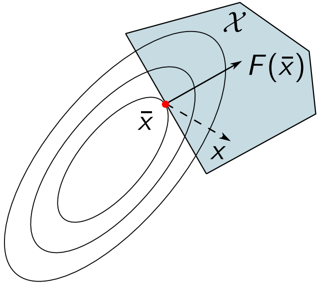

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$(\delta_1, \ldots, \delta_N) \longrightarrow$ Sampled RVI $\longrightarrow$ solution x_S, risk $V(x_S)$

$\rightsquigarrow$ **assume:** existence & uniqueness of solution x_S for all $\{\delta_i\}_{i=1}^N$

First result

For any $\beta \in (0,1)$, $k \in \{0,\ldots,N-1\}$, let $\varepsilon(k)$ be the unique solution of

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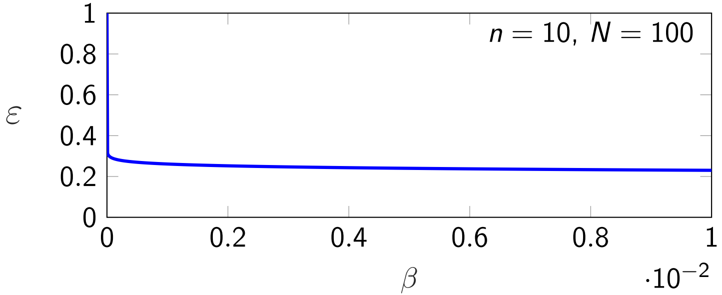

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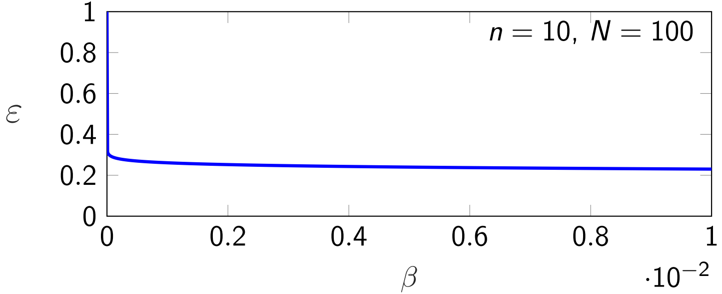

Theorem: assume existence + uniqueness & non-degeneracy

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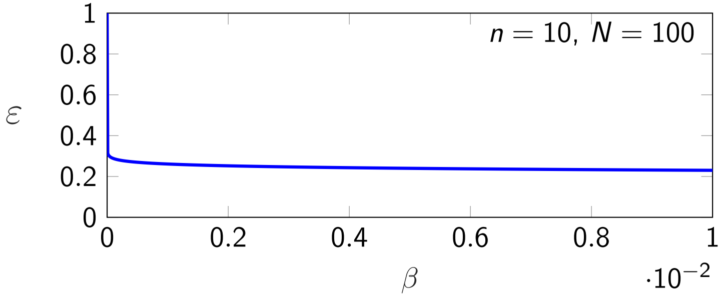

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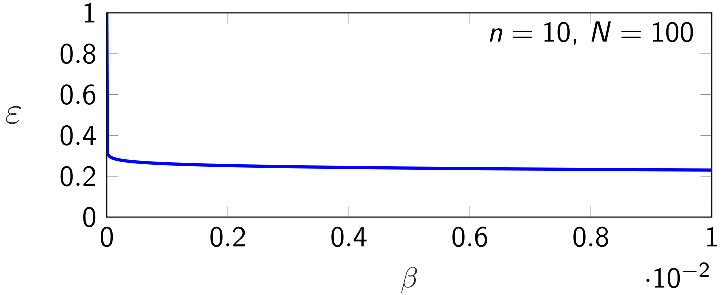

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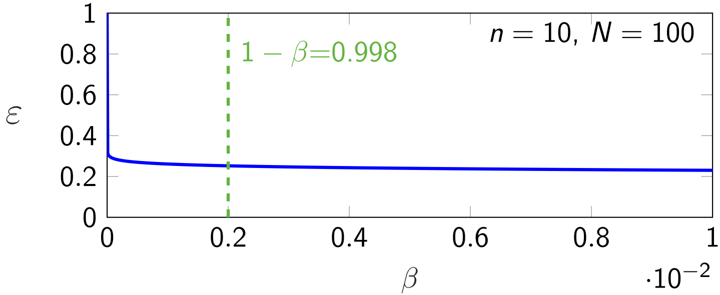

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"with high probability (larger than $1 - \beta$), the risk is small (below ε)"

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Risk: the risk associated to $x \in \mathbb{R}^n$ is $V(x) = \mathbb{P}\{\delta \in \Delta \text{ s.t. } x \notin \mathcal{X}_\delta(x)\}$

The result extends to quasi-variational inequalities

Definition (QVI): given set-valued map $\mathcal{X} : \mathbb{R}^n \rightrightarrows 2^{\mathbb{R}^n}$ and $F : \mathbb{R}^n \to \mathbb{R}^n$,
find $\bar{x} \in \mathcal{X}(\bar{x})$ s.t. $F(\bar{x})^\top (x - \bar{x}) \geq 0, \ \forall\, x \in \mathcal{X}(\bar{x})$

▷ informal: *a VI where the feasible set depends on the point x*

▷ we will use QVI to describe games with uncertain costs

Robust/Sampled QVI: $F : \mathbb{R}^n \to \mathbb{R}^n, \ (\Delta, \mathcal{F}, \mathbb{P}),$ set val. maps $\{\mathcal{X}_\delta\}_{\delta \in \Delta}$

RQVI: find $x_R \in \cap_{\delta\in\Delta} \mathcal{X}_\delta(x_R)$ s.t. $F(x_R)^\top (x - x_R) \geq 0 \quad \forall x \in \cap_{\delta\in\Delta} \mathcal{X}_\delta(x_R)$

S-RQVI: find $x_S \in \cap_{i=1}^N \mathcal{X}_{\delta_i}(x_S)$ s.t. $F(x_S)^\top (x - x_S) \geq 0 \quad \forall x \in \cap_{i=1}^N \mathcal{X}_{\delta_i}(x_S)$
δ_i iid $\sim \mathbb{P}$

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Theorem (informal): the same bounds on the risk hold for QVI.

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Robust NE ([Aghassi and Berstimas]): $x_R \in \mathcal{X}$ is a robust NE if

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Sampled robust NE: $\{\delta_i\}_{i=1}^N$ iid $\sim \mathbb{P}$, $x_S \in \mathcal{X}$ is a sampled robust NE if

$$\max_i C^j(x_S; \delta_i) \leq \max_i C^j(x^j, x_S^{-j}; \delta_i) \qquad \forall x^j \in \mathcal{X}^j, \ \forall j$$

Risk associated to sampled robust NE

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Theorem: existence, uniqueness, non-degeneracy $\Longrightarrow$

- a-priori bound on risk: $\mathbb{P}^N[V^j(x_S) \leq \varepsilon(nM + M)] \geq 1 - \beta$
- a-posteriori bound on risk: $\mathbb{P}^N[V^j(x_S) \leq \varepsilon(s)] \geq 1 - \beta$

Robust Charging games

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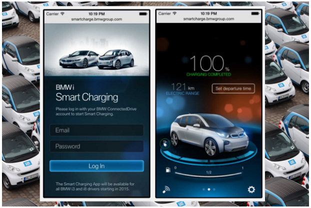

- A fleet of EVs to recharge

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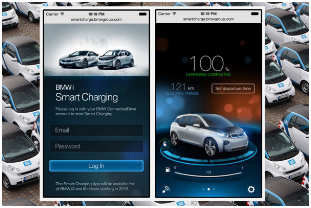

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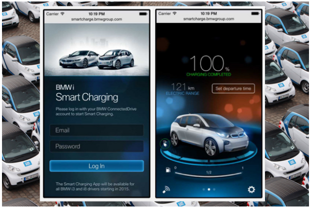

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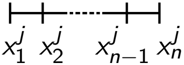
- Each vehicle $\min$ bill in $[1, n]$



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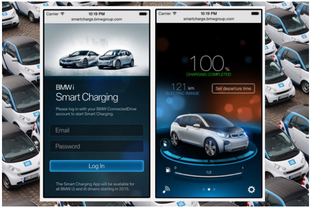

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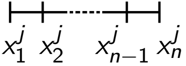

players: $j \in \{1, \dots, M\}$

cost of j: $p(\sum_j x^j + d)^\top x^j$

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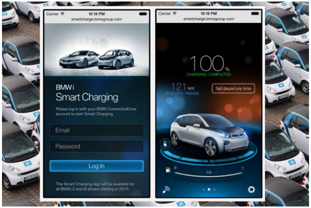

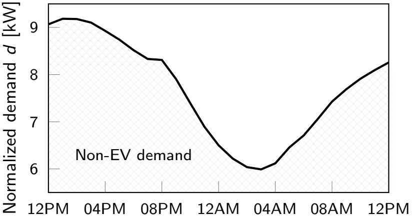

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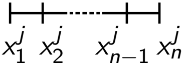

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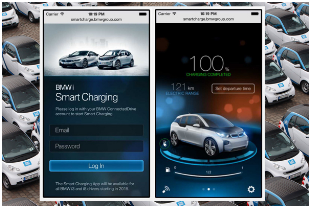

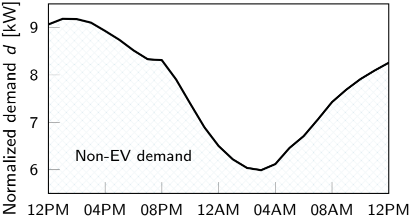

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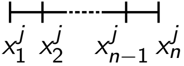

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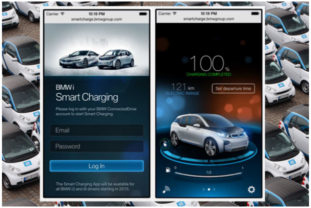

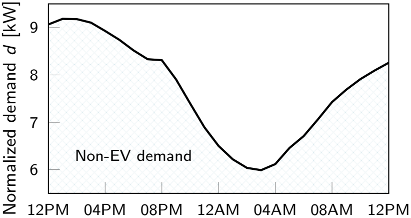

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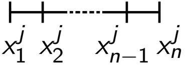

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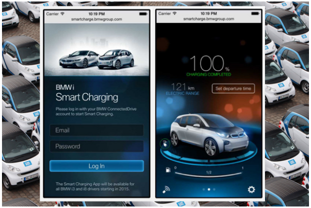

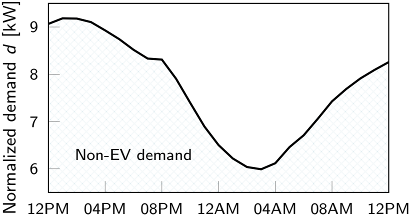

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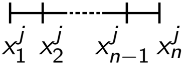

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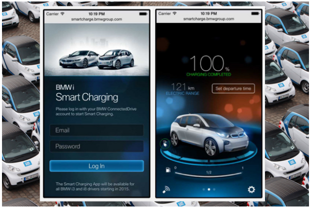

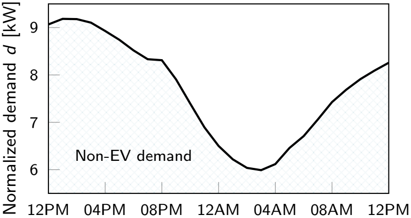

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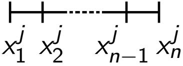

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Q: What guarantees can we provide the users without this assumption?

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Charging profile coordinated to a sampled-robust NE $\rightsquigarrow$ prob. guarantees

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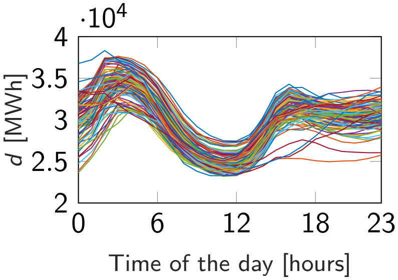

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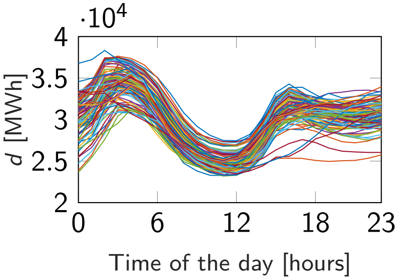

Simulations with $M = 100$ agents, $N = 500$ days of history

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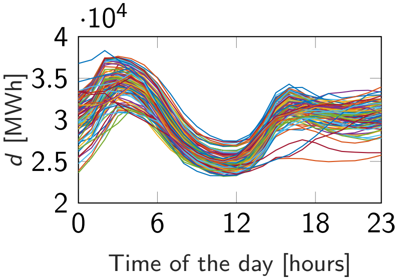

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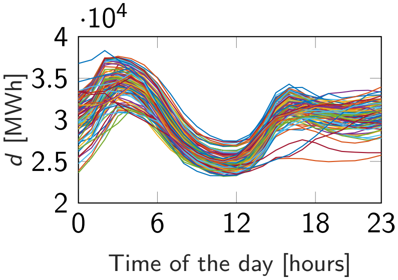

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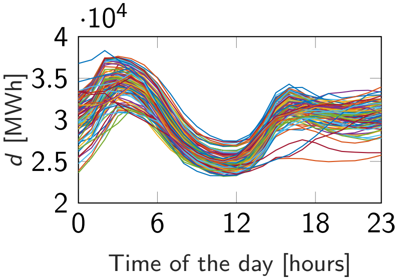

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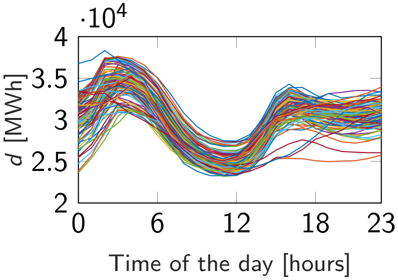

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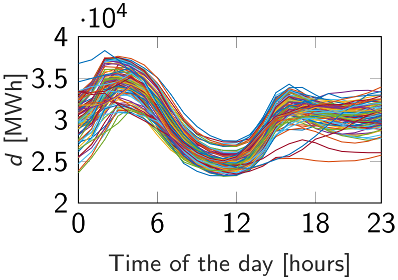

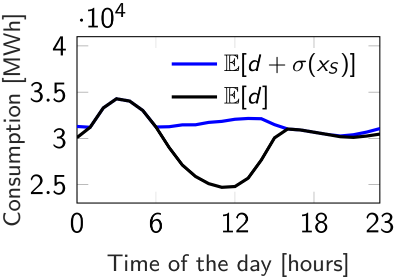

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Conclusions and Outlook

Theme of this talk: take decision based on data and quantify risk

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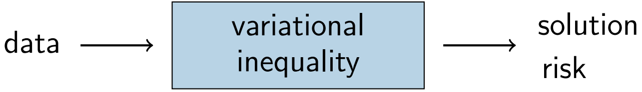

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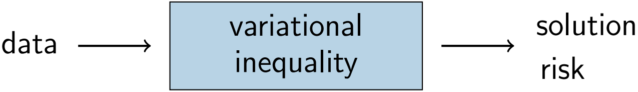

Technical results: - a-priori/a-posteriori bounds for VI and QVI

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data $\longrightarrow$ variational inequality $\longrightarrow$ solution, risk

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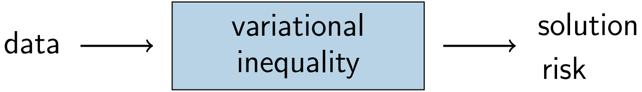

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