# Probabilistic Program Analysis

## Probablistic Abstract Interpretation

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## Approximation and Correctness

Data-flow analyses can be re-formulated in a different scenario where correctness is guaranteed by construction.

Classically, the theory of Abstract Interpretation allows us to

- construct simplified (computable) abstract semantics
- construct approximate solutions
- obtain the correctness of the approximate solution by construction.

# Notions of Approximation

In order theoretic structures we are looking for
Safe Approximations

$$s^* \sqsubseteq s \quad \text{or} \quad s \sqsubseteq s^*$$

In quantitative, vector space structures we want
Close Approximations

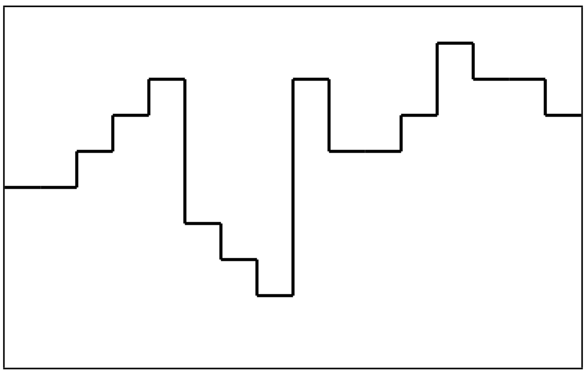
$$\|s - s^*\| = \min_x \|s - x\|$$



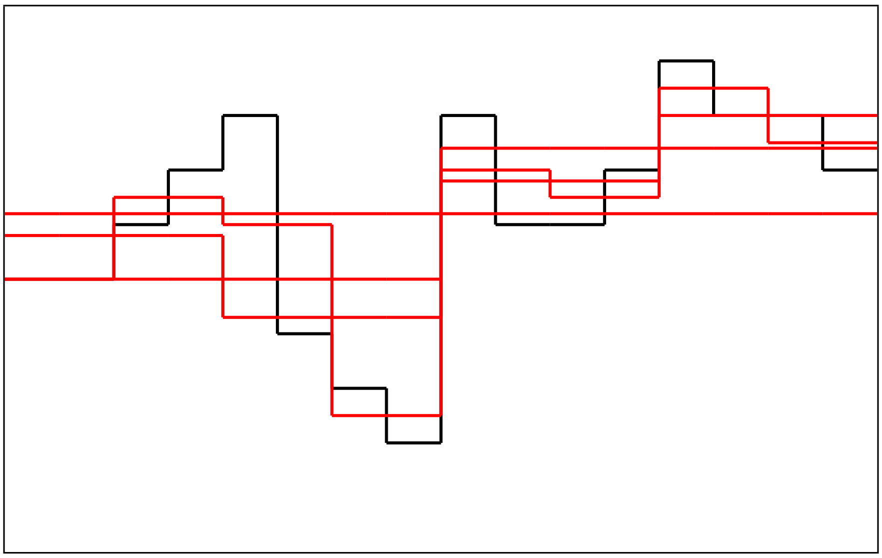


# Example: Function Approximation

Concrete and abstract domain are step-functions on $[a, b]$.
The set of (real-valued) step-function $\mathcal{T}_n$ is based on the sub-division of the interval into $n$ sub-intervals.



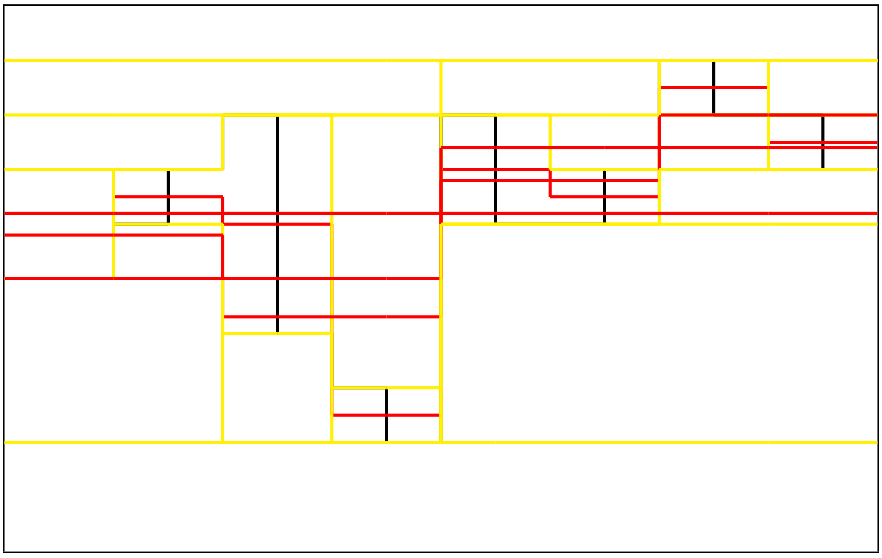
# Close Approximations







# Close vs Correct Approximations



# Abstract Interpretation

Some problems may be have too costly solutions or be uncomputable on a concrete space (complete lattice).
Find abstract descriptions on which computations are easier; then relate the concrete and abstract solutions.

**Definition**

Let $\mathcal{C} = (\mathcal{C}, \leq)$ and $\mathcal{D} = (\mathcal{D}, \sqsubseteq)$ be two partially ordered set. If there are two functions $\alpha : \mathcal{C} \to \mathcal{D}$ and $\gamma : \mathcal{D} \to \mathcal{C}$ such that for all $c \in \mathcal{C}$ and all $d \in \mathcal{D}$:

$$c \leq_{\mathcal{C}} \gamma(d) \text{ iff } \alpha(c) \sqsubseteq d,$$

then $(\mathcal{C}, \alpha, \gamma, \mathcal{D})$ form a Galois connection.



The General Framework
Application to DFA
Abstract Interpretation
The Moore-Penrose Pseudo-Inverse
From Classical to Probabilistic Abstract Interpretation
Probabilistic vs Classical Approximation

# Galois Connections

**Definition**

Let $\mathcal{C} = (\mathcal{C}, \leq_{\mathcal{C}})$ and $\mathcal{D} = (\mathcal{D}, \leq_{\mathcal{D}})$ be two partially ordered sets with two order-preserving functions $\alpha : \mathcal{C} \mapsto \mathcal{D}$ and $\gamma : \mathcal{D} \mapsto \mathcal{C}$. Then $(\mathcal{C}, \alpha, \gamma, \mathcal{D})$ form a Galois connection iff

(i) $\alpha \circ \gamma$ is reductive i.e. $\forall d \in \mathcal{D}$, $\alpha \circ \gamma(d) \leq_{\mathcal{D}} d$,

(ii) $\gamma \circ \alpha$ is extensive i.e. $\forall c \in \mathcal{C}$, $c \leq_{\mathcal{C}} \gamma \circ \alpha(c)$.

**Proposition**

*Let $(\mathcal{C}, \alpha, \gamma, \mathcal{D})$ be a Galois connection. Then $\alpha$ and $\gamma$ are quasi-inverse, i.e.*

(i) $\alpha \circ \gamma \circ \alpha = \alpha$

(ii) $\gamma \circ \alpha \circ \gamma = \gamma$

# General Construction

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "\alpha"] \arrow[d, "f"'] & \mathcal{A}^{\#} \arrow[l, shift left, "\gamma"] \arrow[d, "f^{\#}"] \\ \mathcal{B} \arrow[r, shift left, "\alpha'"] & \mathcal{B}^{\#} \arrow[l, shift left, "\gamma'"] \end{tikzcd}$$

Correct approximation:

$$\alpha' \circ f \leq_{\#} f^{\#} \circ \alpha.$$

Induced semantics:

$$f^{\#} = \alpha \circ f \circ \gamma.$$





# Probabilistic Abstraction Domains

A probabilistic domain is essentially a vector space which represents the distributions $Dist(S)$ on the state space $S$ of a probabilistic transition system, i.e. for finite state spaces

$$\mathcal{V}(S) = \{ (v_s)_{s \in S} \mid v_s \in \mathbb{R} \}.$$

In the finite setting we can identify $\mathcal{V}(S)$ with the Hilbert space $\ell^2(S)$.

The notion of *norm* is essential for our treatment; we will consider normed vector spaces.

# Norm and Operator Norm

A norm on a vector space $\mathcal{V}$ is a map $\|.\| : \mathcal{V} \mapsto \mathbb{R}$ such that for all $v, w \in \mathcal{V}$ and $c \in \mathbb{C}$:

- $\|v\| \geq 0$ ,
- $\|v\| = 0 \Leftrightarrow v = o$,
- $\|cv\| = |c|\|v\|$,
- $\|v + w\| \leq \|v\| + \|w\|$,

with $o \in \mathcal{V}$ the zero vector.

We can always use a norm to define a metric topology on a vector space via the distance function $d(v, w) = \|v - w\|$.

$$\|\mathbf{M}\| = \sup_{v \in \mathcal{V}} \frac{\|\mathbf{M}(v)\|}{\|v\|} = \sup_{\|v\|=1} \|\mathbf{M}(v)\|.$$





# Generalised Inverse

**Definition**

Let $\mathcal{C}$ and $\mathcal{D}$ be two finite-dimensional vector spaces and $\mathbf{A} : \mathcal{C} \rightarrow \mathcal{D}$ a linear map. Then the linear map $\mathbf{A}^\dagger = \mathbf{G} : \mathcal{D} \rightarrow \mathcal{C}$ is the Moore-Penrose pseudo-inverse of $\mathbf{A}$ iff

(i) $\mathbf{A} \circ \mathbf{G} = \mathbf{P}_A$,

(ii) $\mathbf{G} \circ \mathbf{A} = \mathbf{P}_G$,

where $\mathbf{P}_A$ and $\mathbf{P}_G$ denote orthogonal projections onto the ranges of $\mathbf{A}$ and $\mathbf{G}$.

# Least Squares Solutions

**Definition**

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Then $\mathbf{u} \in \mathbb{R}^n$ is called a least squares solution to $\mathbf{Ax} = \mathbf{b}$ if

$$\|\mathbf{Au} - \mathbf{b}\| \leq \|\mathbf{Av} - \mathbf{b}\|, \text{ for all } \mathbf{v} \in \mathbb{R}^n.$$

**Theorem**

*Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Then $\mathbf{A}^\dagger\mathbf{b}$ is the minimal least squares solution to $\mathbf{Ax} = \mathbf{b}$.*





# Orthogonal Projections

**Corollary**

*Let $\mathbf{P}$ be a orthogonal projection on a finite dimensional vector space $\mathcal{V}$. Then for any $\mathbf{x} \in \mathcal{V}$, $\mathbf{Px}$ is the unique closest vector in $\mathcal{V}$ to $\mathbf{x}$ wrt the Euclidean norm.*

# Extraction Functions

An extraction function $\eta : C \mapsto D$ is a mapping from a set of values to their descriptions in $D$.
It is easy to show that

**Proposition**

*Given an extraction function $\eta : \mathcal{C} \mapsto \mathcal{D}$, the quadruple $(\mathcal{P}(\mathcal{C}), \alpha_\eta, \gamma_\eta, \mathcal{P}(\mathcal{D}))$ is a Galois connection with $\alpha_\eta$ and $\gamma_\eta$ defined by:*

$$\alpha_\eta(C') = \{\eta(c) \mid c \in C'\} \text{ and } \gamma_\eta(D') = \{v \mid \eta(v) \in D'\}$$





# Vector Space Lifting

Free vector space construction on a set $S$:

$$\mathcal{V}(S) = \{\sum x_s s \mid x_s \in \mathbb{R}, s \in S\}$$

An obvious way to lift an extraction function to a linear map between vector spaces is to construct the free vector spaces on $\mathcal{C}$ and $\mathcal{D}$ and define:

Vector Space lifting: $\vec{\alpha} : \mathcal{V}(\mathcal{C}) \to \mathcal{V}(\mathcal{D})$

$$\vec{\alpha}(p_1 \cdot \vec{c}_1 + p_2 \cdot \vec{c}_2 + \ldots) = p_i \cdot \eta(c_1) + p_2 \cdot \eta(c_2) \ldots$$

Support Set: **supp** : $\mathcal{V}(\mathcal{C}) \to \mathcal{P}(\mathcal{C})$

$$\textbf{supp}(\vec{x}) = \{c_i \mid \langle c_i, p_i\rangle \in \vec{x} \text{ and } p_i \neq 0\}$$

# Relation with Classical Abstractions

**Lemma**

*Let $\vec{\alpha}$ be a probabilistic abstraction function and let $\vec{\gamma}$ be its Moore-Penrose pseudo-inverse.*

*Then $\vec{\gamma} \circ \vec{\alpha}$ is extensive with respect to the inclusion on the support sets of vectors in $\mathcal{V}(\mathcal{C})$, i.e. $\forall \vec{x} \in \mathcal{V}(\mathcal{C})$,*

$$\mathbf{supp}(\vec{x}) \subseteq \mathbf{supp}(\vec{\gamma} \circ \vec{\alpha}(\vec{x})).$$

Analogously we can show that $\vec{\alpha} \circ \vec{\gamma}$ is reductive. Therefore,

**Proposition**

*$(\vec{\alpha}, \vec{\gamma})$ form a Galois connection wrt the support sets of $\mathcal{V}(\mathcal{C})$ and $\mathcal{V}(\mathcal{D})$, ordered by inclusion.*





# Examples of Lifted Abstractions

Parity Abstraction operator on $\mathcal{V}(\{1, \ldots, n\})$ (with $n$ even):

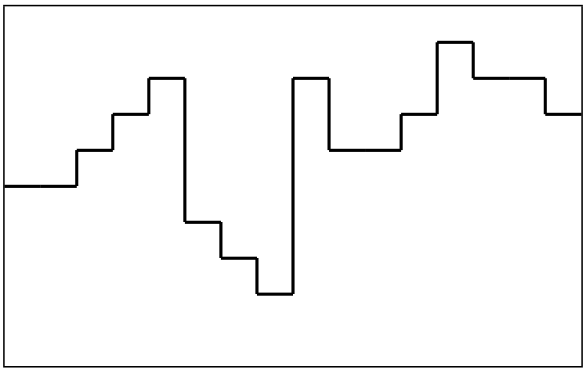
$$\mathbf{A}_p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix} \qquad \mathbf{A}_p^\dagger = \begin{pmatrix} \frac{2}{n} & 0 & \frac{2}{n} & 0 & \ldots & 0 \\ 0 & \frac{2}{n} & 0 & \frac{2}{n} & \ldots & \frac{2}{n} \end{pmatrix}$$

Sign Abstraction operator on $\mathcal{V}(\{-n, \ldots, 0, \ldots, n\})$:

$$\mathbf{A}_s = \begin{pmatrix} 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \vdots & \vdots & \vdots \end{pmatrix} \qquad \mathbf{A}_s^\dagger = \begin{pmatrix} \frac{1}{n} & \ldots & \frac{1}{n} & 0 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & 1 & \ldots & 1 \end{pmatrix}$$

## Example: Function Approximation (ctd.)

Concrete and abstract domain are step-functions on $[a, b]$. The set of (real-valued) step-function $\mathcal{T}_n$ is based on the sub-division of the interval into $n$ sub-intervals.



Each step function in $\mathcal{T}_n$ corresponds to a vector in $\mathbb{R}^n$, e.g.

$$\begin{pmatrix} 5 & 5 & 6 & 7 & 8 & 4 & 3 & 2 & 8 & 6 & 6 & 7 & 9 & 8 & 8 & 7 \end{pmatrix}$$





## Example: Abstraction Matrices

$$\mathbf{A}_8 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

# Approximation Estimates

Compute the *least square error* as

$$\|f - f\mathbf{AG}\|.$$

$$\begin{aligned}
\|f - f\mathbf{A}_8\mathbf{G}_8\| &= 3.5355 \\
\|f - f\mathbf{A}_4\mathbf{G}_4\| &= 5.3151 \\
\|f - f\mathbf{A}_2\mathbf{G}_2\| &= 5.9896 \\
\|f - f\mathbf{A}_1\mathbf{G}_1\| &= 7.6444
\end{aligned}$$





# Concrete Semantics (LOS)

$$\mathbf{T}(P) = \sum_{\langle i, p_{ij}, j\rangle \in \mathit{flow}(P)} p_{ij} \cdot \mathbf{T}(\ell_i, \ell_j),$$

where

$$\mathbf{T}(\ell_i, \ell_j) = \mathbf{N} \otimes \mathbf{E}(\ell_i, \ell_j),$$

with **N** an operator representing a state update while the second factor realises the transfer of control from label $\ell_i$ to label $\ell_j$.

## Abstract Semantics

Moore-Penrose Pseudo-Inverse of a Tensor Product is:

$$(\mathbf{A}_1 \otimes \mathbf{A}_2 \otimes \ldots \otimes \mathbf{A}_n)^\dagger = \mathbf{A}_1^\dagger \otimes \mathbf{A}_2^\dagger \otimes \ldots \otimes \mathbf{A}_n^\dagger$$

Via linearity we can construct $\mathbf{T}^\#$ in the same way as $\mathbf{T}$, i.e

$$\mathbf{T}^\#(P) = \sum_{\langle i, p_{ij}, j\rangle \in \mathcal{F}(P)} p_{ij} \cdot \mathbf{T}^\#(\ell_i, \ell_j)$$

with local abstraction of individual variables:

$$\mathbf{T}^\#(\ell_i, \ell_j) = (\mathbf{A}_1^\dagger \mathbf{N}_{i1} \mathbf{A}_1) \otimes (\mathbf{A}_2^\dagger \mathbf{N}_{i2} \mathbf{A}_2) \otimes \ldots \otimes (\mathbf{A}_v^\dagger \mathbf{N}_{iv} \mathbf{A}_v) \otimes \mathbf{M}_{ij}$$





## Argument

$$\begin{aligned}
\mathbf{T}^\# &= \mathbf{A}^\dagger \mathbf{T} \mathbf{A} \\
&= \mathbf{A}^\dagger (\sum_{i,j} \mathbf{T}(i,j)) \mathbf{A} \\
&= \sum_{i,j} \mathbf{A}^\dagger \mathbf{T}(i,j) \mathbf{A} \\
&= \sum_{i,j} (\bigotimes_k \mathbf{A}_k)^\dagger \mathbf{T}(i,j) (\bigotimes_k \mathbf{A}_k) \\
&= \sum_{i,j} (\bigotimes_k \mathbf{A}_k)^\dagger (\bigotimes_k \mathbf{N}_{ik}) (\bigotimes_k \mathbf{A}_k) \\
&= \sum_{i,j} \bigotimes_k (\mathbf{A}_k^\dagger \mathbf{N}_{ik} \mathbf{A}_k)
\end{aligned}$$

# Parity Analysis

Determine at each program point whether a variable is *even* or *odd*.

Parity Abstraction operator on $\mathcal{V}(\{0,\ldots,n\})$ (with $n$ even):

$$\mathbf{A}_p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix} \qquad \mathbf{A}^\dagger = \begin{pmatrix} \frac{2}{n} & 0 & \frac{2}{n} & 0 & \ldots & 0 \\ 0 & \frac{2}{n} & 0 & \frac{2}{n} & \ldots & \frac{2}{n} \end{pmatrix}$$





# Example

1: $[m \leftarrow i]^1$;
2: **while** $[n > 1]^2$ **do**
3: $\quad [m \leftarrow m \times n]^3$;
4: $\quad [n \leftarrow n - 1]^4$
5: **od**
6: $[\textbf{stop}]^5$

$$\begin{aligned} \mathbf{T} &= \mathbf{U}(\mathtt{m} \leftarrow i) \otimes \mathbf{E}(1,2) \\ &+ \mathbf{P}(n > 1) \otimes \mathbf{E}(2,3) \\ &+ \mathbf{P}(n \leq 1) \otimes \mathbf{E}(2,5) \\ &+ \mathbf{U}(\mathtt{m} \leftarrow m \times n) \otimes \mathbf{E}(3,4) \\ &+ \mathbf{U}(\mathtt{n} \leftarrow n - 1) \otimes \mathbf{E}(4,2) \\ &+ \mathbf{I} \otimes \mathbf{E}(5,5) \end{aligned}$$

$$\begin{aligned} \mathbf{T}^\# &= \mathbf{U}^\#(\mathtt{m} \\ &+ \mathbf{P}^\#(n \\ &+ \mathbf{P}^\#(n \\ &+ \mathbf{U}^\#(\mathtt{m} \\ &+ \mathbf{U}^\#(\mathtt{n} \\ &+ \mathbf{I}^\# \otimes \mathbf{E} \end{aligned}$$

# Abstract Semantics

Abstraction: $\mathbf{A} = \mathbf{A}_p \otimes \mathbf{I}$, i.e. $m$ abstract (parity) but $n$ concrete.

$$\begin{aligned}
\mathbf{T}^{\#} &= \mathbf{U}^{\#}(m \leftarrow 1) \otimes \mathbf{E}(1,2) \\
&+ \mathbf{P}^{\#}(n > 1) \otimes \mathbf{E}(2,3) \\
&+ \mathbf{P}^{\#}(n \leq 1) \otimes \mathbf{E}(2,5) \\
&+ \mathbf{U}^{\#}(m \leftarrow m \times n) \otimes \mathbf{E}(3,4) \\
&+ \mathbf{U}^{\#}(n \leftarrow n - 1) \otimes \mathbf{E}(4,2) \\
&+ \mathbf{I}^{\#} \otimes \mathbf{E}(5,5)
\end{aligned}$$

$$\mathbf{U}^{\#}(m \leftarrow 1) =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & 0 & \ldots & 0 \end{pmatrix}$$



$$= \begin{pmatrix} 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & 1 & \ldots & 0 \end{pmatrix}$$



# Implementation

Implementation of concrete and abstract semantics of Factorial using **octave**. Ranges: $n \in \{1, \ldots, d\}$ and $m \in \{1, \ldots, d!\}$.

| $d$ | $\dim(\mathbf{T}(F))$ | $\dim(\mathbf{T}^{\#}(F))$ |
|---|---|---|
| 2 | 45 | 30 |
| 3 | 140 | 40 |
| 4 | 625 | 50 |
| 5 | 3630 | 60 |
| 6 | 25235 | 70 |
| 7 | 201640 | 80 |
| 8 | 1814445 | 90 |
| 9 | 18144050 | 100 |

Using uniform initial distributions $\mathbf{d_0}$ for $n$ and $m$.

# Scalablity

The abstract probabilities for $m$ being **even** or **odd** when we execute the abstract program for various $d$ values are:

| $d$ | **even** | **odd** |
|---|---|---|
| 10 | 0.81818 | 0.18182 |
| 100 | 0.98019 | 0.019802 |
| 1000 | 0.99800 | 0.0019980 |
| 10000 | 0.99980 | 0.00019998 |





# Live Variable Analysis

1: $[\textbf{skip}]^1 [y \leftarrow 2 \times x]^1$
2: **if** $[odd(y)]^2$ **then**
3: $\quad [x \leftarrow x + 1]^3$
4: **else**
5: $\quad [y \leftarrow y + 1]^4$
6: **fi**
7: $[y \leftarrow y + 1]^5$

Classical Analysis: $\mathrm{LV}_{entry}(2) = \{x, y\}$

Probabilistic Analysis: $\mathrm{LV}_{entry}(2) = \{\langle x, \frac{1}{2}\rangle, \langle y, 1\rangle\}$
$\mathrm{LV}_{entry}(2) = \quad \mathrm{LV}_{entry}(2) = \{\langle y, 1\rangle\}$

# Program "Transformation"

1: $[y \leftarrow 2 \times x]^1$
2: **if** $[odd(y)]^2$ **then**
3: $\quad [x \leftarrow x + 1]^3$
4: **else**
5: $\quad [y \leftarrow y + 1]^4$
6: **fi**
7: $[y \leftarrow y + 1]^5$

1: $[y \leftarrow 2 \times x]^1$
2: $[\textbf{choose}]^2$
3: $\quad p_\top : [x \leftarrow x + 1]^3$
4: **or**
5: $\quad p_\perp : [y \leftarrow y + 1]^4$
6: $[y \leftarrow y + 1]^5$

Determine branching probabilities in a first-phase analysis and utilise this information to perform the actual analysis:

$$p^\top = \mathbf{A}^\dagger \cdot \mathbf{P}(b = \textbf{true}) \cdot \mathbf{A} \text{ and } p^\perp = \mathbf{A}^\dagger \cdot \mathbf{P}(b = \textbf{false}) \cdot \mathbf{A}$$





# Syntax of pWhile with Pointers

$$
\begin{array}{rcl}
S & ::= & [\textbf{skip}]^\ell \\
 & \mid & [\textbf{stop}]^\ell \\
 & \mid & [p \leftarrow e]^\ell \\
 & \mid & S_1;\ S_2 \\
 & \mid & [\textbf{choose}]^\ell\ p_1 : S_1 \ \textbf{or}\ p_2 : S_2 \\
 & \mid & \textbf{if}\ [b]^\ell\ \textbf{then}\ S_1\ \textbf{else}\ S_2 \\
 & \mid & \textbf{while}\ [b]^\ell\ \textbf{do}\ S
\end{array}
$$

$$p \ ::= \ *^r\mathtt{x} \text{ with } \mathtt{x} \in \textbf{Var} \qquad e \ ::= \ a \mid b \mid l$$

$$a \ ::= \ n \mid p \mid a_1 \odot a_2 \qquad l \ ::= \ \mathrm{NIL} \mid p \mid \&p$$

$$b \ ::= \ \textbf{true} \mid \textbf{false} \mid p \mid \neg b \mid b_1 \between b_2 \mid a_1 \gtrless a_2$$

# Example

**if** $[(\mathtt{z}_0 \bmod 2 = 0)]^1$ **then**
  $[\mathtt{x} \leftarrow \&\mathtt{z}_1]^2;\ [\mathtt{y} \leftarrow \&\mathtt{z}_2]^3$
**else**
  $[\mathtt{x} \leftarrow \&\mathtt{z}_2]^4;\ [\mathtt{y} \leftarrow \&\mathtt{z}_1]^5$
**fi**
$[\textbf{stop}]^6$

$[\textbf{choose}]^1$
  $\frac{1}{2} : ([\mathtt{x} \leftarrow \&\mathtt{z}_1]^2;\ [\mathtt{y} \leftarrow \&\mathtt{z}_2]^3)$
**or**
  $\frac{1}{2} : ([\mathtt{x} \leftarrow \&\mathtt{z}_2]^4;\ [\mathtt{y} \leftarrow \&\mathtt{z}_1]^5)$
$[\textbf{stop}]^6$





# Test Operators and Filters

Select a certain value $c \in \textbf{Value}$:

$$(\mathbf{P}(c))_{ij} = \begin{cases} 1 & \text{if } i = c = j \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbf{P}(2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Select a certain classical state $\sigma \in \textbf{State}$:

$$\mathbf{P}(\sigma) = \bigotimes_{i=1}^{v} \mathbf{P}(\sigma(\mathtt{x}_i))$$

$$\mathbf{P}(\sigma(\mathtt{x}_1 \mapsto 2, \mathtt{x}_2 \mapsto 4)) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \text{[illegible]} \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \text{[illegible]} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

## Selection via Projections

Filtering out *relevant* configurations, i.e. only those which fulfill a certain condition. Use diagonal matrix **P**:

$$(\mathbf{P})_{ii} = \begin{cases} 1 & \text{if condition holds for } c_i \in \textbf{Value} \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{pmatrix}^t \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ d_2 \\ d_3 \\ 0 \\ d_5 \\ 0 \end{pmatrix}^t$$





## Example

**if** $[(\mathtt{z}_0 \bmod 2 = 0)]^1$ **then**
$[\mathtt{x} \leftarrow \&\mathtt{z}_1]^2;\ [\mathtt{y} \leftarrow \&\mathtt{z}_2]^3$
**else**
$[\mathtt{x} \leftarrow \&\mathtt{z}_2]^4;\ [\mathtt{y} \leftarrow \&\mathtt{z}_1]^5$
**fi**
$[\textbf{stop}]^6$

$\textbf{Var} = \{\mathtt{x}, \mathtt{y}, \mathtt{z}_0, \mathtt{z}_1, \mathtt{z}_2\}$

$$\mathbf{P}(\mathtt{z}_0 \bmod 2 = 0) = \mathbf{I} \otimes \mathbf{I} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \otimes \mathbf{I} \otimes \mathbf{I}$$

# Update Operators

For all initial values change to constant $c \in$ **Value**:

$$(\mathbf{U}(c))_{ij} = \begin{cases} 1 & \text{if } j = c \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbf{U}(3) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Set value of variable $\mathtt{x}_k \in$ **Var** to constant $c \in$ **Value**:

$$\mathbf{U}(\mathtt{x}_k \leftarrow c) = \left(\bigotimes_{i=1}^{k-1} \mathbf{I}\right) \otimes \mathbf{U}(c) \otimes \left(\bigotimes_{i=k+1}^{v} \mathbf{I}\right)$$



Set variable $\mathtt{x}_k \in$ **Var** to value given by expression $e = a \mid b \mid l$:



# Update for Pointers

For an assignment with a pointer on the l.h.s. we need to determine recursevly the actual variable $p$ is pointing to:

$$\mathbf{U}(*^r \mathtt{x}_k \leftarrow e) = \sum_{\mathtt{x}_i} \mathbf{P}(\mathtt{x}_k = \&\mathtt{x}_i)\mathbf{U}(*^{r-1}\mathtt{x}_i \leftarrow e)$$

Note that we always get eventually to the base case, i.e. where $p$ refers to a concrete variable $\mathtt{x}_k$ and thus the update operator $\mathbf{U}(\mathtt{x}_k \leftarrow e)$ from before.
For a pointer of second order with $\mathtt{x}_2 \rightarrow \mathtt{x}_1 \rightarrow \mathtt{x}_0$ we get:

$$\begin{aligned} \mathbf{U}(**\mathtt{x}_2 \leftarrow 4) &= \sum_{\mathtt{x}_i} \mathbf{P}(\mathtt{x}_2 = \&\mathtt{x}_i)\mathbf{U}(*\mathtt{x}_i \leftarrow 4) \\ \mathbf{U}(*\mathtt{x}_1 \leftarrow 4) &= \sum \mathbf{P}(\mathtt{x}_1 = \&\mathtt{x}_i)\mathbf{U}(\mathtt{x}_i \leftarrow 4) \end{aligned}$$



$\mathbf{U}(\mathtt{x}_0 \leftarrow 4)$

# Example

**if** $[(\mathtt{z}_0 \text{ mod } 2 = 0)]^1$ **then**
$[\mathtt{x} \leftarrow \&\mathtt{z}_1]^2;\ [\mathtt{y} \leftarrow \&\mathtt{z}_2]^3$
**else**
$[\mathtt{x} \leftarrow \&\mathtt{z}_2]^4;\ [\mathtt{y} \leftarrow \&\mathtt{z}_1]^5$
**fi**
$[\textbf{stop}]^6$

$$\begin{aligned}
&\mathbf{P}(\mathit{even}(\mathtt{z}_0)) \otimes \mathbf{E}(1,2)+\\
&\mathbf{P}(\mathit{odd}(\mathtt{z}_0)) \otimes \mathbf{E}(1,4)+\\
&\mathbf{U}(\mathtt{x} \leftarrow \&\mathtt{z}_1) \otimes \mathbf{E}(2,3)+\\
&\mathbf{U}(\mathtt{y} \leftarrow \&\mathtt{z}_2) \otimes \mathbf{E}(3,6)+\\
&\mathbf{U}(\mathtt{x} \leftarrow \&\mathtt{z}_2) \otimes \mathbf{E}(4,5)+\\
&\mathbf{U}(\mathtt{y} \leftarrow \&\mathtt{z}_1) \otimes \mathbf{E}(5,6)+\\
&\mathbf{I} \otimes \mathbf{E}(6,6)
\end{aligned}$$

$[\textbf{choose}]^1$
$\frac{1}{2} : ([\mathtt{x} \leftarrow \&\mathtt{z}_1]^2;\ [\mathtt{y} \leftarrow \&\mathtt{z}_2]^3)$
**or**
$\frac{1}{2} : ([\mathtt{x} \leftarrow \&\mathtt{z}_2]^4;\ [\mathtt{y} \leftarrow \&\mathtt{z}_1]^5)$
$[\textbf{stop}]^6$

$$\begin{aligned}
&\tfrac{1}{2} \cdot (\mathbf{I} \otimes \mathbf{E}(1,2))+\\
&\tfrac{1}{2} \cdot (\mathbf{I} \otimes \mathbf{E}(1,4))+\\
&\mathbf{U}(\mathtt{x} \leftarrow \&\mathtt{z}_1) \otimes \mathbf{E}(2,3)+\\
&\mathbf{U}(\mathtt{y} \leftarrow \&\mathtt{z}_2) \otimes \mathbf{E}(3,6)+\\
&\mathbf{U}(\mathtt{x} \leftarrow \&\mathtt{z}_2) \otimes \mathbf{E}(4,5)+\\
&\mathbf{U}(\mathtt{y} \leftarrow \&\mathtt{z}_1) \otimes \mathbf{E}(5,6)+\\
&\mathbf{I} \otimes \mathbf{E}(6,6)
\end{aligned}$$





# Abstract Branching Probabilities

The abstract tests $\mathbf{P}^\#$ describe the branching probabilities depending on abstract values.

For example, consider $\mathbf{P}(n)$ testing if a variable with values $1, \ldots, n$ is a prime number.



$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

## Transforming **if** into **choose**

Based on the abstract branching probabilities we can replace tests, e.g. in **if**'s, by probabilistic choices. In a a first phase, we need to determine the probabilities of abstract values.

If we have the probabilities of $\mathrm{z}_0$ being even or odd we can compute the probabilities of the **then** and **else** branch using $\mathbf{P}^\#$. For $\mathrm{z}_0$ being even and odd with the same probability:

**if** $[(\mathrm{z}_0 \bmod 2 = 0)]^1$ **then**
  $[\mathrm{x} \leftarrow \&\mathrm{z}_1]^2;\ [\mathrm{y} \leftarrow \&\mathrm{z}_2]^3$
**else**
  $[\mathrm{x} \leftarrow \&\mathrm{z}_2]^4;\ [\mathrm{y} \leftarrow \&\mathrm{z}_1]^5$
**fi**
$[$**stop**$]^6$





## Probabilistic Pointer Analysis

The typical result of a probabilistic pointer analysis is a so-called points-to matrix: records for every program point the probability that a pointer refers to particular (other) variable.

Consider again our standard example.

**if** $[(\mathrm{z}_0 \bmod 2 = 0)]^1$ **then**
  $[\mathrm{x} \leftarrow \&\mathrm{z}_1]^2;\ [\mathrm{y} \leftarrow \&\mathrm{z}_2]^3$
**else**
  $[\mathrm{x} \leftarrow \&\mathrm{z}_2]^4;\ [\mathrm{y} \leftarrow \&\mathrm{z}_1]^5$
**fi**
$[$**stop**$]^6$

Where do $\mathrm{x}$ and $\mathrm{y}$ point to with what probabilities?

# Points-To Matrix vs Points-To Tensor

**if** $[(\mathtt{z}_0 \text{ mod } 2 = 0)]^1$ **then**
    $[\mathtt{x} \leftarrow \&\mathtt{z}_1]^2$; $[\mathtt{y} \leftarrow \&\mathtt{z}_2]^3$
**else**
    $[\mathtt{x} \leftarrow \&\mathtt{z}_2]^4$; $[\mathtt{y} \leftarrow \&\mathtt{z}_1]^5$
**fi**
$[\textbf{stop}]^6$

## Points-To Matrix

| | $\&\mathtt{x}$ | $\&\mathtt{y}$ | $\&\mathtt{z}_0$ | $\&\mathtt{z}_1$ | $\&\mathtt{z}_2$ |
|---|---|---|---|---|---|
| $\mathtt{x}$ | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $\mathtt{y}$ | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ |

## Points-To Matrix

$$(0,0,0,\frac{1}{2},\frac{1}{2}) \quad - \quad (0,0,0,\frac{1}{2},\frac{1}{2})$$



## Points-To Tensor

$$\frac{1}{2} \cdot (0,0,0,1,0) \otimes (0,0,0,0,1) + \frac{1}{2} \cdot (0,0,0,0,1) \otimes (0,0,0,1,0)$$