Program Analysis (70020)
Correctness of an Analysis

Herbert Wiklicky

Department of Computing
Imperial College London

herbert@doc.ic.ac.uk
h.wiklicky@imperial.ac.uk

Autumn 2024

Correctness

Questions: Is a program analysis correct? Are the results reflecting what is really happening when the program is run?

In other words: What is the relation between the (concrete) semantics of a program, i.e. the transition relation $\Rightarrow$ and/or its transitive closure $\Rightarrow^*$, and the (solutions to) the program analysis $Analysis_\circ$ and $Analysis_\bullet$.

For example: Is a variable LV identifies as 'live' indeed useful, or more importantly, is a 'non-live' variable really 'dead', i.e. is it save to eliminate it (at least locally).

Syntax of WHILE

The labelled syntax of the language WHILE is given by the following **abstract syntax**:

$$a \quad ::= \quad x \mid n \mid a_1 \; op_a \; a_2$$

$$b \quad ::= \quad true \mid false \mid not \; b \mid b_1 \; op_b \; b_2 \mid a_1 \; op_r \; a_2$$

$$S \quad ::= \quad [x := a]^\ell$$
$$\mid [\textbf{skip}]^\ell$$
$$\mid S_1 ; S_2$$
$$\mid \textbf{if } [b]^\ell \textbf{ then } S_1 \textbf{ else } S_2$$
$$\mid \textbf{while } [b]^\ell \textbf{ do } S$$

Sketches of a Formal Semantics

Memory is modelled by an abstract state, i.e. functions of type

$$\textbf{State} \quad = \quad \textbf{Var} \rightarrow \textbf{Z}.$$

For boolean and arithmetic expressions we assume that we know what they "evaluate to" in a state $s \in \textbf{State}$.   Then the semantics for **AExp** is a *total* function

$$[\![\; \cdot \;]\!]_{\mathcal{A}} \; \cdot \quad : \quad \textbf{AExp} \rightarrow \textbf{State} \rightarrow \textbf{Z}$$

and the semantics of boolean expressions is given by

$$[\![\; \cdot \;]\!]_{\mathcal{B}} \; \cdot \quad : \quad \textbf{BExp} \rightarrow \textbf{State} \rightarrow \{ \textbf{tt}, \textbf{ff} \}$$

Evaluating Expressions

Let us look at a program with two variables $\textbf{Var} = \{x, y\}$.
Two possible states in this case could be for example:

$$s_0 = [x \mapsto 0, y \mapsto 1] \text{ and } s_1 = [x \mapsto 1, y \mapsto 1]$$

We can evaluate an expression like $x + y \in \textbf{AExp}$:

$$
\begin{aligned}
[\![x + y]\!]_{\mathcal{A}} s_0 &= 0 + 1 = 1 \\
[\![x + y]\!]_{\mathcal{A}} s_1 &= 1 + 1 = 2
\end{aligned}
$$

or a Boolean expression like $x + y \leq 1 \in \textbf{BExp}$:

$$
\begin{aligned}
[\![x + y \leq 1]\!]_{\mathcal{B}} s_0 &= 1 \leq 1 = \textbf{tt} \\
[\![x + y \leq 1]\!]_{\mathcal{B}} s_1 &= 2 \leq 1 = \textbf{ff}
\end{aligned}
$$

Execution and Transitions

The configurations describe the current state of the execution.

$$
\begin{aligned}
\langle S, s \rangle &\quad \ldots \quad S \text{ is to be executed in state } s, \\
s &\quad \ldots \quad \text{a terminal state (i.e. } \langle ., s \rangle \text{)}.
\end{aligned}
$$

The transition relation $\Rightarrow$ specify the (possible) computational steps during the execution starting from a certain configuration

$$\langle S, s \rangle \Rightarrow \langle S', s' \rangle$$

and at the end of the computation

$$\langle S, s \rangle \Rightarrow s'$$

Execution Rules (SOS) [Provided in Exam]

(ass) $\langle [\texttt{x}\texttt{:=}a]^{\ell}, s\rangle \Rightarrow s[x \mapsto [\![a]\!]_{\mathcal{A}}s]$

(skip) $\langle [\textbf{skip}]^{\ell}, s\rangle \Rightarrow s$

(sq^1) $\dfrac{\langle S_1, s\rangle \Rightarrow \langle S_1', s'\rangle}{\langle S_1\texttt{;}S_2, s\rangle \Rightarrow \langle S_1'\texttt{;}S_2, s'\rangle}$

(sq^T) $\dfrac{\langle S_1, s\rangle \Rightarrow s'}{\langle S_1\texttt{;}S_2, s\rangle \Rightarrow \langle S_2, s'\rangle}$

(if^T) $\langle \textbf{if } [b]^{\ell} \textbf{ then } S_1 \textbf{ else } S_2, s\rangle \Rightarrow \langle S_1, s\rangle$ if $[\![b]\!]_{\mathcal{B}}s = \textbf{tt}$

(if^F) $\langle \textbf{if } [b]^{\ell} \textbf{ then } S_1 \textbf{ else } S_2, s\rangle \Rightarrow \langle S_2, s\rangle$ if $[\![b]\!]_{\mathcal{B}}s = \textbf{ff}$

(wh^T) $\langle \textbf{while } [b]^{\ell} \textbf{ do } S, s\rangle \Rightarrow \langle S\textbf{; while } [b]^{\ell} \textbf{ do } S, s\rangle$ if $[\![b]\!]_{\mathcal{B}}s = \textbf{tt}$

(wh^F) $\langle \textbf{while } [b]^{\ell} \textbf{ do } S, s\rangle \Rightarrow s$ if $[\![b]\!]_{\mathcal{B}}s = \textbf{ff}$

A SOS Example

Consider a (perhaps rather vacuous) program like:

$$S \equiv [z := x + y]^{\ell}\texttt{;}\textbf{while } [true]^{\ell'} \textbf{ do } [\textbf{ skip }]^{\ell''}$$

$s_0 = [x \mapsto 0, y \mapsto 1, z \mapsto 0]$ and $s_1 = [x \mapsto 0, y \mapsto 1, z \mapsto 1]$

Then $\langle S, s_0\rangle$ executes as follows:

$$\begin{aligned}
\langle S, s_0\rangle \quad &\Rightarrow \quad \langle \textbf{while } [true]^{\ell'} \textbf{ do } [\textbf{ skip }]^{\ell''}, s_1\rangle \\
&\Rightarrow \quad \langle [\textbf{ skip }]^{\ell''}\texttt{;}\textbf{while } [true]^{\ell'} \textbf{ do } [\textbf{ skip }]^{\ell''}, s_1\rangle \\
&\Rightarrow \quad \langle \textbf{while } [true]^{\ell'} \textbf{ do } [\textbf{ skip }]^{\ell''}, s_1\rangle \\
&\Rightarrow \quad \langle [\textbf{ skip }]^{\ell''}\texttt{;}\textbf{while } [true]^{\ell'} \textbf{ do } [\textbf{ skip }]^{\ell''}, s_1\rangle \\
&\Rightarrow \quad \ldots
\end{aligned}$$

Lemma 1

(i) If $\langle S, s \rangle \Rightarrow s'$ then
$\mathit{final}(S) = \{\mathit{init}(S)\}$.

(ii) If $\langle S, s \rangle \Rightarrow \langle S', s' \rangle$ then
$\mathit{final}(S) \supseteq \mathit{final}(S')$.

(iii) If $\langle S, s \rangle \Rightarrow \langle S', s' \rangle$ then
$\mathit{flow}(S) \supseteq \mathit{flow}(S')$.

(iv) If $\langle S, s \rangle \Rightarrow \langle S', s' \rangle$ then
$\mathit{blocks}(S) \supseteq \mathit{blocks}(S')$.

(v) If $\langle S, s \rangle \Rightarrow \langle S', s' \rangle$ then
S label consistent implies S' label consistent.

Lemma 1 - Proof (i) [Not for Exam]

Proof.
The proof is by induction on the shape of the inference tree.
Consider the only three non-vacuous cases:

(ass): $\langle [\texttt{x:=}a]^\ell, s \rangle \Rightarrow s[x \mapsto [\![a]\!]s]$

$$\mathit{final}([\texttt{x:=}a]^\ell) = \{\ell\} = \{\mathit{init}([\texttt{x:=}a]^\ell)\}.$$

(skip): $\langle [\texttt{skip}]^\ell, s \rangle \Rightarrow s$

$$\mathit{final}([\texttt{skip}]^\ell) = \{\ell\} = \{\mathit{init}([\texttt{skip}]^\ell)\}.$$

(wh^F): $\langle \texttt{while } [b]^\ell \texttt{ do } S, s \rangle \Rightarrow s$ with $[\![b]\!] = \textbf{false}$

$$\mathit{final}(\texttt{while } [b]^\ell \texttt{ do } S) = \{\ell\} = \{\mathit{init}(\texttt{while } [b]^\ell \texttt{ do } S)\}.$$

Lemma 1 - Proof (ii) [Not for Exam]

Proof (cont).

(seq^1): $\langle S_1 ; S_2, s \rangle \Rightarrow \langle S_1' ; S_2, s' \rangle$ because
$\langle S_1, s \rangle \Rightarrow \langle S_1', s' \rangle$:

$$final(S_1 ; S_2) = final(S_2) = final(S_1' ; S_2).$$

(seq^T): ...

(if^T): $\langle \text{if } [b]^\ell \text{ then } S_1 \text{ else } S_2, s \rangle \Rightarrow \langle S_1, s \rangle$
with $[\![b]\!] = $ **true**:

$$final(\text{if } [b]^\ell \text{ then } S_1 \text{ else } S_2) = \\ final(S_1) \cup final(S_2) \supseteq final(S_1).$$

(if^F): ...

(wh^T): ...

$\square$

LV Equations: *LV$^=$*

The Live Variable Analysis is given as the solution to the
following system of equations:

$$\mathsf{LV}_{exit}(\ell) = \begin{cases} \emptyset, \text{if } \ell \in final(S_\star) \\ \bigcup \{\mathsf{LV}_{entry}(\ell') \mid (\ell', \ell) \in flow^R(S_\star)\}, \text{otherwise} \end{cases}$$

$$\mathsf{LV}_{entry}(\ell) = (\mathsf{LV}_{exit}(\ell) \backslash kill_{\mathsf{LV}}([B]^\ell) \cup gen_{\mathsf{LV}}([B]^\ell) \\ \text{where } [B]^\ell \in blocks(S_\star)$$

Solutions via Iteration Operator

$$\mathsf{LV}_{entry}(1) \quad = \quad \mathbf{F}_1^\bullet(\mathsf{LV}_{entry}(1), \ldots, \mathsf{LV}_{exit}(n))$$

$$\cdots \quad \cdots \quad \cdots$$

$$\mathsf{LV}_{entry}(n) \quad = \quad \mathbf{F}_n^\bullet(\mathsf{LV}_{entry}(1), \ldots, \mathsf{LV}_{exit}(n))$$

$$\mathsf{LV}_{exit}(1) \quad = \quad \mathbf{F}_1^\circ(\mathsf{LV}_{entry}(1), \ldots, \mathsf{LV}_{exit}(n))$$

$$\cdots \quad \cdots \quad \cdots$$

$$\mathsf{LV}_{exit}(n) \quad = \quad \mathbf{F}_n^\circ(\mathsf{LV}_{entry}(1), \ldots, \mathsf{LV}_{exit}(n))$$

becomes a function on the lattice $\mathcal{P}(\mathbf{Var})^{2n}$

$$\mathbf{F} : \mathcal{P}(\mathbf{Var})^{2n} \to \mathcal{P}(\mathbf{Var})^{2n}$$

$$\mathbf{F}_i^\bullet(\mathsf{LV}_{entry}(1), \ldots, \mathsf{LV}_{exit}(n)) \quad = \quad \mathsf{LV}_{entry}(i)$$

$$\mathbf{F}_i^\circ(\mathsf{LV}_{entry}(1), \ldots, \mathsf{LV}_{exit}(n)) \quad = \quad \mathsf{LV}_{exit}(i)$$

LV Constraints: $LV^{\subseteq}$

The Live Variable Analysis is equivalently given as the solution to the following system of constraints:

$$\mathsf{LV}_{exit}(\ell) \quad \supseteq \quad \begin{cases} \emptyset, \text{if } \ell \in \textit{final}(S_\star) \\ \bigcup\{\mathsf{LV}_{entry}(\ell') \mid (\ell', \ell) \in \textit{flow}^R(S_\star)\}, \text{otherwise} \end{cases}$$

$$\mathsf{LV}_{entry}(\ell) \quad \supseteq \quad (\mathsf{LV}_{exit}(\ell) \backslash \textit{kill}_{\mathsf{LV}}([B]^\ell) \cup \textit{gen}_{\mathsf{LV}}([B]^\ell)$$
$$\text{where } [B]^\ell \in \textit{blocks}(S_\star)$$

LV Solutions to *LV*$^=$ and *LV*$^\subseteq$

Consider collections *live* $=$ (*live*$_{\text{entry}}$, *live*$_{\text{exit}}$) of functions:

$$live_{\text{entry}} : \mathbf{Lab}_\star \to \mathcal{P}(\mathbf{Var}_\star)$$
$$live_{\text{exit}} : \mathbf{Lab}_\star \to \mathcal{P}(\mathbf{Var}_\star)$$

If *live* solves *LV*$^=$ for a statement *S* we write:

$$live \models LV^=(S)$$

If *live* solves *LV*$^\subseteq$ for a statement *S* we write:

$$live \models LV^\subseteq(S)$$

Theorem 1

Given a label consistent program $S_\star$.

If
- *live* $\models LV^=(S_\star)$

then
- *live* $\models LV^\subseteq(S_\star)$.

That is: The least solution of $LV^=(S_\star)$ coincides with the least solution to $LV^\subseteq(S_\star)$.

Theorem 1 - Proof [Not for Exam]

Proof.

If $live \models LV^=(S_\star)$ also $live \models LV^\subseteq(S_\star)$ as "$\supseteq$" includes "$=$".

To show that $LV^=(S_\star)$ and $LV^\subseteq(S_\star)$ have the same least solution consider the iteration operator $\mathbf{F} = \mathbf{F}_{LV} = \mathbf{F}_{LV}^S$

$$live \models LV^\subseteq(S_\star) \quad \text{iff} \quad live \sqsupseteq \mathbf{F}(live)$$
$$live \models LV^=(S_\star) \quad \text{iff} \quad live = \mathbf{F}(live)$$

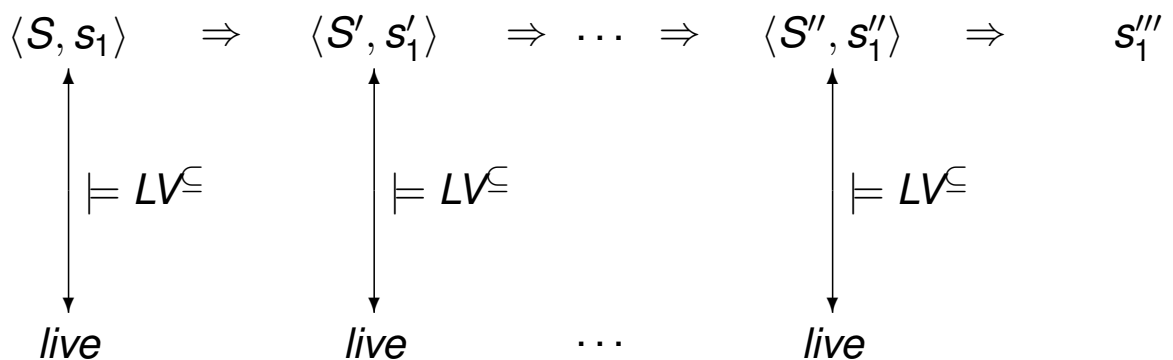By Tarski's Fixed Point Theorem we have:

$$lfp(\mathbf{F}) = \bigsqcap\{live \mid live \sqsupseteq \mathbf{F}(live)\} = \bigsqcap\{live \mid live = \mathbf{F}(live)\}.$$

Since $lfp(\mathbf{F}) = \mathbf{F}(lfp(\mathbf{F}))$ and $lfp(\mathbf{F}) \sqsupseteq \mathbf{F}(lfp(\mathbf{F}))$ we see that we get the same least solutions. $\quad\square$

Preservation of Solution

During the (actual) execution of any program $S_\star$ a solution to the Live Variable analysis $LV^\subseteq(S_\star)$ remains a solution.

$$\langle S, s_1\rangle \quad \Rightarrow \quad \langle S', s_1'\rangle \quad \Rightarrow \quad \cdots \quad \Rightarrow \quad \langle S'', s_1''\rangle \quad \Rightarrow \quad s_1'''$$

$$\models LV^\subseteq \qquad\qquad \models LV^\subseteq \qquad\qquad\qquad \models LV^\subseteq$$

$$live \qquad\qquad\qquad live \qquad\qquad \cdots \qquad\qquad live$$

Lemma 2

Given a label consistent program S_1.

If

- ▶ $live \models LV^{\subseteq}(S_1)$ and
- ▶ $flow(S_1) \supseteq flow(S_2)$ and
- ▶ $blocks(S_1) \supseteq blocks(S_2)$

then

- ▶ $live \models LV^{\subseteq}(S_2)$

with S_2 being label consistent.

Proof [Not for Exam].

If S_1 is label consistent and $blocks(S_1) \supseteq blocks(S_2)$ then S_2 is also label consistent.

If $live \models LV^{\subseteq}(S_1)$ then $live$ also satisfy each constraint in $LV^{\subseteq}(S_2)$ and hence $live \models LV^{\subseteq}(S_2)$. □

Lemma 3

Given a label consistent program S.

If

- ▶ $live \models LV^{\subseteq}(S)$ and
- ▶ $\langle S, s \rangle \Rightarrow \langle S', s' \rangle$

then

- ▶ $live \models LV^{\subseteq}(S')$.

Proof [Not for Exam].

Follows directly from Lemma 1 and Lemma 2. □

Lemma 4

Given a label consistent program S.

If
- $live \models LV^{\subseteq}(S)$

then for all $(\ell, \ell') \in flow(S)$ we have:
- $live_{exit}(\ell) \supseteq live_{entry}(\ell')$

Proof [Not for Exam].
Follows immediately from the construction of $LV^{\subseteq}(S)$. □

Correctness Relation

Assume that V is a set of *live variables*.

Define the correctness relation via

$$s_1 \sim_V s_2 \text{ iff } \forall x \in V : s_1(x) = s_2(x).$$

In other word:
Two states are equivalent iff for all live variables – i.e. all "practical purposes" – the states s_1 and s_2 agree on the variables in V.

Example

Consider $[x := y + z]^\ell$ and $V_1 = \{y, z\}$ and $V_2 = \{x\}$.

$$s_1 \sim_{V_1} s_2 \text{ means } s_1(y) = s_2(y) \wedge s_1(z) = s_2(z).$$

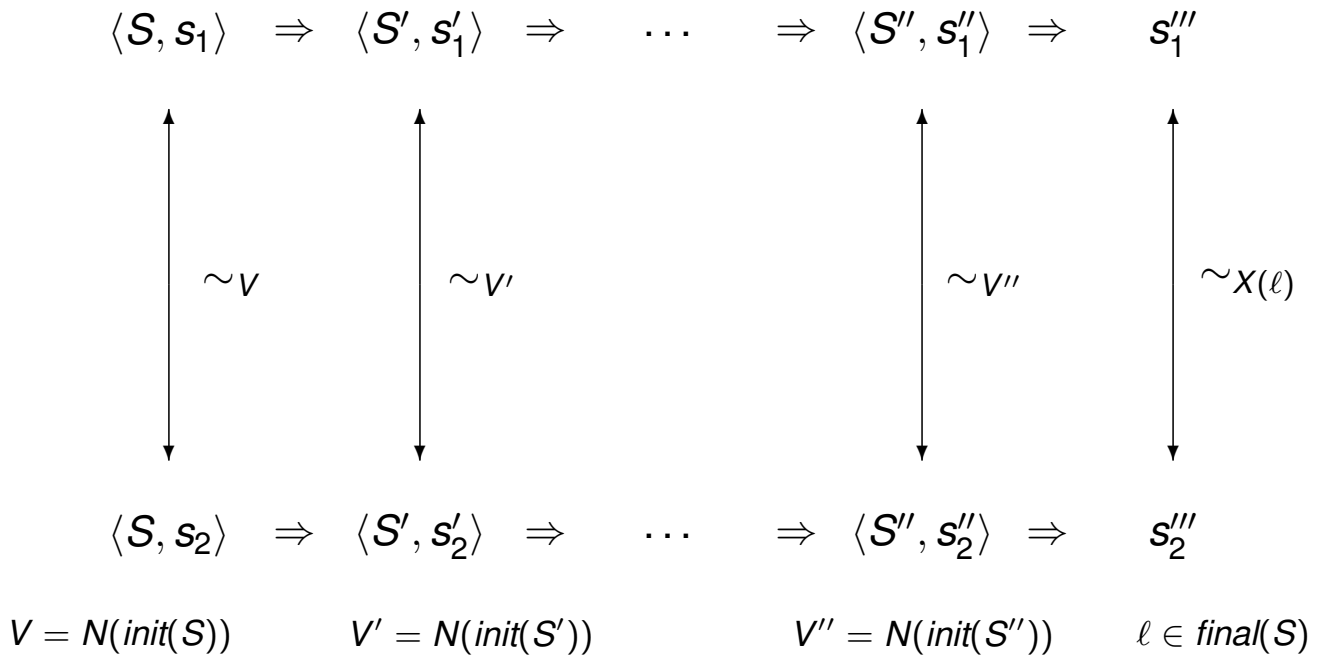$$s_1 \sim_{V_2} s_2 \text{ means } s_1(x) = s_2(x).$$

Assume $\langle [x := y + z]^\ell, s_1 \rangle \Rightarrow s_1'$, $\langle [x := y + z]^\ell, s_2 \rangle \Rightarrow s_2'$ then

$$s_1 \sim_{V_1} s_2 \text{ ensures } s_1' \sim_{V_2} s_2'.$$

If $V_2 = \mathsf{LV}_{exit}(\ell)$ thus is the set of live variables after $[x := y + z]^\ell$ then $V_1 = \mathsf{LV}_{entry}(\ell)$ is the set of live variables before $[x := y + z]^\ell$.

Correctness of *LV* Analysis

$$\langle S, s_1 \rangle \Rightarrow \langle S', s_1' \rangle \Rightarrow \quad \cdots \quad \Rightarrow \langle S'', s_1'' \rangle \Rightarrow s_1'''$$

$\sim_V \qquad \sim_{V'} \qquad\qquad \sim_{V''} \qquad \sim_{X(\ell)}$

$$\langle S, s_2 \rangle \Rightarrow \langle S', s_2' \rangle \Rightarrow \quad \cdots \quad \Rightarrow \langle S'', s_2'' \rangle \Rightarrow s_2'''$$

$V = N(init(S)) \qquad V' = N(init(S')) \qquad V'' = N(init(S'')) \qquad \ell \in final(S)$

Short-hand notation: $N(\ell) = live_{\text{entry}}(\ell)$ and $X(\ell) = live_{\text{exit}}(\ell)$.

Lemma 5

Given a label consistent program S.

If

- $live \models LV^{\subseteq}(S)$

then

- $s_1 \sim live_{exit(\ell)} \; s_2$ implies $s_1 \sim live_{entry(\ell')} \; s_2$ for all $(\ell, \ell') \in flow(S)$.

Proof [Not for Exam].
Follows directly from Lemma 4 and the definition of $\sim_V$. $\quad\square$

Theorem 2

Given a label consistent program S.

If

- $live \models LV^{\subseteq}(S)$

then

(i) If $\langle S, s_1 \rangle \Rightarrow \langle S', s_1' \rangle$ and $s_1 \sim live_{entry(init(S))} \; s_2$ then there exists s_2' such that $\langle S, s_2 \rangle \Rightarrow \langle S', s_2' \rangle$ and $s_1' \sim live_{entry(init(S'))} \; s_2'$.

(ii) If $\langle S, s_1 \rangle \Rightarrow s_1'$ and $s_1 \sim live_{entry(init(S))} \; s_2$ then there exists s_2' such that $\langle S, s_2 \rangle \Rightarrow s_2'$ and $s_1' \sim live_{exit(init(S))} \; s_2'$.

Theorem 2 - Proof [Not for Exam]

Proof.
The proof is by induction on the shape of the inference tree.

$$(\text{ass}):\ \ldots$$
$$(\text{skip}):\ \ldots$$
$$(\text{seq}^1):\ \ldots$$
$$(\text{seq}^T):\ \ldots$$
$$(\text{if}^T):\ \ldots$$
$$(\text{if}^F):\ \ldots$$
$$(\text{wh}^T):\ \ldots$$
$$(\text{wh}^F):\ \ldots$$

Theorem 2 - Proof (ass) [Not for Exam]

Proof (cont).
The proof is by induction on the shape of the inference tree.

(ass): We have $\langle [\mathrm{x}\,\mathtt{:=}\,a]^\ell, s_1 \rangle \Rightarrow s_1[x \mapsto [\![a]\!] s_1]$ and from the specification of the constraints:

$$live_{\text{entry}}(\ell) = (live_{\text{exit}}(\ell) \backslash \{x\}) \cup FV(a)$$

and therefore

$$s_1 \sim_{live_{\text{entry}}(\ell)} s_2 \text{ implies } [\![a]\!](s_1) = [\![a]\!](s_2)$$

because the value of a depends only on variables in it.

Thus with $s_2' = s_2[x \mapsto [\![a]\!]_{\mathcal{A}} s_2]$ we have $s_1'(x) = s_2'(x)$ and thus $s_1' \sim_{live_{\text{exit}}(\ell)} s_2'$.

Theorem 2 - Proof (skip) [Not for Exam]

Proof (cont).

(skip): We have $\langle [\texttt{skip}]^\ell, s_1 \rangle \Rightarrow s_1$ and from the specification of the constraints we get:

$$live_{\text{entry}}(\ell) = (live_{\text{exit}}(\ell)\backslash\emptyset) \cup \emptyset = live_{\text{exit}}(\ell)$$

Thus taking s_2' to be s_2 we get $s_1' \sim_{live_{\text{exit}}(\ell)} s_2'$ as required.

Theorem 2 - Proof (seq^1) [Not for Exam]

Proof (cont).

(seq^1): We have $\langle S_1 ; S_2, s_1 \rangle \Rightarrow \langle S_1' ; S_2, s_1' \rangle$ because of $\langle S_1, s_1 \rangle \Rightarrow \langle S_1', s_1' \rangle$.

By construction we have $flow(S_1 ; S_2) \supseteq flow(S_1)$ and also $blocks(S_1 ; S_2) \supseteq blocks(S_1)$, thus by Lemma 2 $live \models LV^{\subseteq}(S_1)$ and by the induction hypothesis there exists a s_2' such that

$$\langle S_1, s_2 \rangle \Rightarrow \langle S_1', s_2' \rangle \text{ and } s_1' \sim_{live_{\text{entry}}(init(S_1'))} s_2'$$

and the result follows.

Theorem 2 - Proof (seq^T) [Not for Exam]

Proof (cont).

(seq^T): We have $\langle S_1; S_2, s_1 \rangle \Rightarrow \langle S_2, s_1' \rangle$ because of $\langle S_1, s_1 \rangle \Rightarrow s_1'$. Again by Lemma 2, *live* is a solution to $LV^{\subseteq}(S_1)$ and thus by induction hypothesis there exists a s_2' such that

$$\langle S_1, s_2 \rangle \Rightarrow s_2' \text{ and } s_1' \sim_{live_{exit}(init(S_1))} s_2'$$

Now we have:

$$\{(\ell, init(S_2)) \mid \ell \in final(S_1)\} \subseteq flow(S_1; S_2)$$

and by Lemma 1, $final(S_1) = \{init(S_1)\}$. Thus by Lemma 5

$$s_1' \sim_{live_{entry}(init(S_2))} s_2'$$

and the result follows.

Theorem 2 - Proof (if^T) & (if^F) [Not for Exam]

Proof (cont).

(if^T): We have $\langle \texttt{if } [b]^\ell \texttt{ then } S_1 \texttt{ else } S_2, s_1 \rangle \Rightarrow \langle S_1, s_1 \rangle$ with $[\![b]\!](s_1) = \textbf{true}$.
Since $s_1 \sim_{live_{entry}(\ell)} s_2$ and $live_{entry}(\ell) \supseteq FV(b)$ we also have $[\![b]\!](s_2) = \textbf{true}$ (the value of b is only dependent on the variables occurring in it) and thus

$$\langle \texttt{if } [b]^\ell \texttt{ then } S_1 \texttt{ else } S_2, s_2 \rangle \Rightarrow \langle S_1, s_2 \rangle$$

From the constraints we get $live_{entry}(\ell) \supseteq live_{exit}(\ell)$ and hence $s_1 \sim_{live_{exit}(\ell)} s_2$.
Since $(\ell, init(S_1)) \in flow(S)$ Lemma 5 gives $s_1 \sim_{live_{entry}(init(S_1))} s_2$ as required.

(if^F): similar to case (if^T).

Theorem 2 - Proof (wh^T) [Not for Exam]

Proof (cont).

(wh^T): $\langle \texttt{while } [b]^\ell \texttt{ do } S, s_1 \rangle \Rightarrow \langle S; \texttt{ while } [b]^\ell \texttt{ do } S, s_1 \rangle$
with $[\![b]\!](s_1) = \textbf{true}$.
Since $s_1 \sim_{live_{\text{entry}}(\ell)} s_2$ and $live_{\text{entry}}(\ell) \supseteq FV(b)$ we
also have $[\![b]\!](s_2) = \textbf{true}$ and thus

$$\langle \texttt{while } [b]^\ell \texttt{ do } S, s_2 \rangle \Rightarrow \langle S; \texttt{ while } [b]^\ell \texttt{ do } S, s_2 \rangle$$

Again since $live_{\text{entry}}(\ell) \supseteq live_{\text{exit}}(\ell)$ we have
$s_1 \sim_{live_{\text{exit}}(\ell)} s_2$ and then

$$s_1 \sim_{live_{\text{entry}}(init(S))} s_2$$

from Lemma 5 as
$(\ell, init(S)) \in flow(\texttt{while } [b]^\ell \texttt{ do } S)$.

Theorem 2 - Proof (wh^F) [Not for Exam]

Proof (cont).

(wh^F): We have $\langle \texttt{while } [b]^\ell \texttt{ do } S, s_1 \rangle \Rightarrow s_1$ with
$[\![b]\!](s_1) = \textbf{false}$.

Since $s_1 \sim_{live_{\text{entry}}(\ell)} s_2$ and $live_{\text{entry}}(\ell) \supseteq FV(b)$ and
we also have $[\![b]\!](s_2) = \textbf{false}$ and thus:

$$\langle \texttt{while } [b]^\ell \texttt{ do } S, s_2 \rangle \Rightarrow s_2.$$

From the specification of $LV^\subseteq$ we have
$live_{\text{entry}}(\ell) \supseteq live_{\text{exit}}(\ell)$ and thus $s_1 \sim_{live_{\text{exit}}(\ell)} s_2$.

$\square$

Corollary 1

Given a label consistent program S.

If
- $live \models LV^{\subseteq}(S)$

then

    (i) If $\langle S, s_1 \rangle \Rightarrow^* \langle S', s_1' \rangle$ and $s_1 \sim_{live_{entry}(init(S))} s_2$ then there exists s_2' such that $\langle S, s_2 \rangle \Rightarrow^* \langle S', s_2' \rangle$ and $s_1' \sim_{live_{entry}(init(S'))} s_2'$.

    (ii) If $\langle S, s_1 \rangle \Rightarrow^* s_1'$ and $s_1 \sim_{live_{entry}(init(S))} s_2$ then there exists s_2' such that $\langle S, s_2 \rangle \Rightarrow^* s_2'$ and $s_1' \sim_{live_{exit}(\ell)} s_2'$ for some $\ell \in final(S)$.

Proof [Not for Exam].

The proof is by induction on the length of the derivation sequences and uses Theorem 2.   ☐