Quantum Computation (CO484)

Quantum States and Evolution

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Quantum Postulates

- The state of an (isolated) quantum system is represented by a (normalised) vector in a complex Hilbert space $\mathcal{H}$.
- An observable is represented by a self-adjoint matrix (operator) $\mathbf{A}$ acting on the Hilbert space $\mathcal{H}$.
- The expected result (average) when measuring observable $\mathbf{A}$ of a system in state $|x\rangle \in \mathcal{H}$ is given by:

$$\langle A \rangle_x = \langle x| \mathbf{A} |x\rangle = \langle x| |\mathbf{A}x\rangle$$

- The only possible results are eigen-values λ_i of $\mathbf{A}$.
- The probability of measuring λ_n in state $|x\rangle$ is given by:

$$Pr(A = \lambda_n | x) = \langle x| \mathbf{P}_n |x\rangle = \langle x| |\mathbf{P}_n x\rangle$$

with $\mathbf{P}_n = |\lambda_n\rangle\langle\lambda_n|$ the orthogonal projection onto the space generated by eigen-vector $|\lambda_n\rangle = |n\rangle$ of $\mathbf{A}$.

Complex Numbers

Quantitative information, e.g. measurement results, is usually represented by real numbers $\mathbb{R}$. For quantum systems we need to consider also complex numbers $\mathbb{C}$.

A complex number $z \in \mathbb{C}$ is a (formal) combinations of two reals $x, y \in \mathbb{R}$:

$$z = x + iy$$

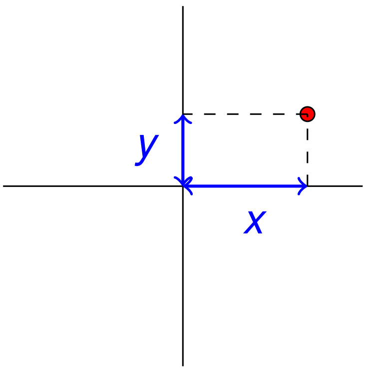
with $i^2 = -1$ or $i = \sqrt{-1}$. The complex conjugate of a complex number $z = x + iy \in \mathbb{C}$ is:

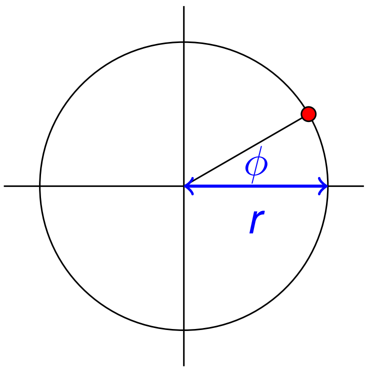
$$z^* = \overline{z} = \overline{x + iy} = x - iy = z^\dagger$$

Hauptsatz of Algebra
Complex numbers are algebraically closed: Every polynomial of order n over $\mathbb{C}$ has exactly n roots.

Polar Coordinates

One can represent numbers $z \in \mathbb{C}$ using the complex plane.





Conversion:

$$x = r \cdot \cos(\phi) \qquad y = r \cdot \sin(\phi)$$

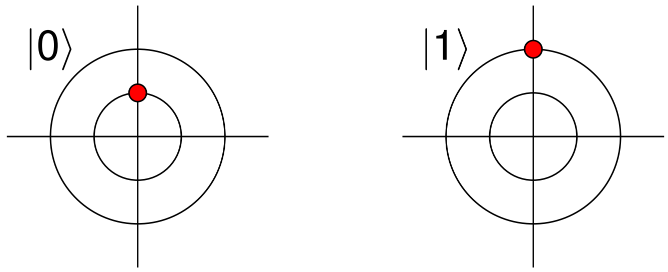
$$r = \sqrt{x^2 + y^2} \qquad \phi = \arctan(\frac{y}{x})$$

Another representation:

$$(r, \phi) = r \cdot e^{i\phi} \qquad e^{i\phi} = \cos(\phi) + i\sin(\phi),$$

Computational Quantum States

Consider a simple systems with two **degrees of freedom**.



Definition

A qubit (quantum bit) is a quantum state of the form

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

where α and β are complex numbers with $|\alpha|^2 + |\beta|^2 = 1$.

Qubits live in a two-dimensional complex vector, more precisely, Hilbert space $\mathbb{C}^2$ and are **normalised**, i.e. $\|\,|\psi\rangle\,\| = \langle\psi \mid \psi\rangle = 1$.

Vector Spaces

A **Vector Space** (over a field $\mathbb{K}$, e.g. $\mathbb{R}$ or $\mathbb{C}$) is a set $\mathcal{V}$ together with two operations:

Scalar Product $\;.\cdot.: \mathbb{K} \times \mathcal{V} \mapsto \mathcal{V}$

Vector Addition $\;.+.: \mathcal{V} \times \mathcal{V} \mapsto \mathcal{V}$

such that $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{K}$:

1. $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$
2. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
3. $\exists \mathbf{o} : \mathbf{x} + \mathbf{o} = \mathbf{x}$
4. $\exists -\mathbf{x} : \mathbf{x} + (-\mathbf{x}) = \mathbf{o}$

5. $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$
6. $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$
7. $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$
8. $1\mathbf{x} = \mathbf{x}$ $(1 \in \mathbb{K})$

Tuple Spaces

Theorem

All finite dimensional vector spaces are isomorphic to the (finite) Cartesian product of the underlying field $\mathbb{K}^n$ (i.e. $\mathbb{R}^n$ or $\mathbb{C}^n$).

$$\vec{x} = (x_1, x_2, x_3, \dots, x_n) \text{ represents } \mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{b}_i$$

$$\vec{y} = (y_1, y_2, y_3, \dots, y_n) \text{ represents } \mathbf{y} = \sum_{i=1}^{n} y_i \mathbf{b}_i$$

Finite dimensional vectors can be represented as tuples via their coordinates with respect to a base $\{\mathbf{b}_i\}_{i=1}^n$.

$$\alpha\vec{x} = (\alpha x_1, \alpha x_2, \alpha x_3, \dots, \alpha x_n)$$

$$\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3, \dots, x_n + y_n)$$

Hilbert Spaces

A complex vector space $\mathcal{H}$ is called an **Inner Product Space** or **(Pre-)**Hilbert Space if there is a complex valued function $\langle .,. \rangle$ on $\mathcal{H} \times \mathcal{H}$ that satisfies $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{H}$ and $\forall \alpha \in \mathbb{C}$:

1. $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$
2. $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \iff \mathbf{x} = \mathbf{o}$
3. $\langle \alpha\mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$
4. $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$
5. $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$

The function $\langle .,. \rangle$ is called an inner product on $\mathcal{H}$.

Caveat: Linear in first or second argument?

Mathematical Convention:

$$\langle \alpha \mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$$

Physical Convention:

$$\langle \mathbf{x} \mid \alpha \mathbf{y} \rangle = \alpha \langle \mathbf{x} \mid \mathbf{y} \rangle$$

In mathematics we have:

$$\langle \mathbf{x}, \alpha \mathbf{y} \rangle = \overline{\langle \alpha \mathbf{y}, \mathbf{x} \rangle} = \bar{\alpha} \overline{\langle \mathbf{y}, \mathbf{x} \rangle} = \bar{\alpha} \langle \mathbf{x}, \mathbf{y} \rangle$$

For physicists it is simply:

$$\langle \mathbf{x} \mid \alpha \mathbf{y} \rangle = \alpha \langle \mathbf{x} \mid \mathbf{y} \rangle$$

Basis Vectors

A set of vectors $\mathbf{x}_i$ is said to be linearly independent iff

$$\sum \lambda_i \mathbf{x}_i = \mathbf{o} \quad \text{implies that} \quad \forall\, i : \lambda_i = 0$$

Two vectors in a Hilbert space are orthogonal iff

$$\langle \mathbf{x}, \mathbf{y} \rangle = 0$$

An orthonormal system in a Hilbert space is a set of linearly independent set of vectors with:

$$\langle \mathbf{b}_i, \mathbf{b}_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{iff } i = j \\ 0 & \text{iff } i \neq j \end{cases}$$

Theorem

For a Hilbert space there exists an orthonormal basis $\{\mathbf{b}\}$. The representation of each vector is unique:

$$\mathbf{x} = \sum_i x_i \mathbf{b}_i = \sum_i \langle \mathbf{x}, \mathbf{b}_i \rangle \mathbf{b}_i$$

The Finite-Dimensional Hilbert Spaces $\mathbb{C}^n$

We represent vectors and their transpose using coordinates:

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = |x\rangle\,, \quad \vec{y} = (y_1, \ldots, y_n) = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}^T = \langle y|$$

The adjoint of $\vec{x} = (x_1, \ldots, x_n)$ is given by

$$\vec{x}^\dagger = (\bar{x}_1, \ldots, \bar{x}_n)^T = (x_1^*, \ldots, x_n^*)^T$$

The inner product is then represented by:

$$\langle \vec{y}, \vec{x} \rangle = \sum_i \bar{y}_i x_i = \sum_i y_i^* x_i$$

We can also define a norm (length) $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$.

Dual and Adjoint States

A linear functional on a vector space $\mathcal{V}$ is a map $f : \mathcal{V} \to \mathbb{K}$ such that (i) $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$ and (ii) $f(\alpha\mathbf{x}) = \alpha f(\mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}, \alpha \in \mathbb{K}$.
The space of all linear functionals on $\mathcal{V}$ form the dual space $\mathcal{V}^*$.

Theorem (Riesz Representation Theorem)

Every linear functional $f : \mathcal{H} \to \mathbb{C}$ on a Hilbert space $\mathcal{H}$ can be represented by a vector $\mathbf{y}_f$ in $\mathcal{H}$, such that

$$f(\mathbf{x}) = \langle \mathbf{y}_f, \mathbf{x} \rangle = f_y(\mathbf{x})$$

Dual Hilbert spaces $\mathcal{H}^*$ are isomorphic to the original Hilbert space $\mathcal{H}^*$; in particular we have: $(\mathbb{C}^n)^* = \mathbb{C}^n$.

We represent vectors or ket-vectors as column vectors; and functionals, dual vector or bra-vectors as row vectors.

Dirac Notation and Einstein Convention

We will use throughout P.A.M. Dirac's bra-(c)-ket notation:

$$\langle \mathbf{x}_i, \mathbf{y}_j \rangle = \langle \vec{x}_i, \vec{y}_j \rangle \text{ denoted as } \langle x_i | \, | y_j \rangle = \langle i | \, | j \rangle$$

We will enumerate the (eigen-)base vectors (of an operator):

$$\vec{b}_i = \mathbf{b}_i \text{ or } \vec{e}_i = \mathbf{e}_i \quad \text{are denoted by} \quad | i \rangle$$

but we may need also to specify the coordinates of a vector:

- Ket-Vectors (column): $|x\rangle = (x_j)_{j=1}^n$ in $\mathbb{C}^n$.
- Bra-Vectors (row): $\langle x| = (x^j)_{j=1}^n$ in $(\mathbb{C}^n)^* = \mathbb{C}^n$.

A. Einstein: If in an expression there are matching sub- and super-scripts then this implicitely indicates a summation,

$$\bar{x}_i y^i = \sum_i \bar{x}_i y^i = \langle \vec{x}, \vec{y} \rangle \text{ and } x_i y^{i*} = \sum_i x_i \bar{y}^i = \langle \vec{x} \mid \vec{y} \rangle$$

Qubit

The postulates of **Quantum Mechanics** simply require that a computational quantum state is represented by a normalised vector in $\mathbb{C}^n$. A qubit is a two-dimensional quantum state in $\mathbb{C}^2$

We represent the **coordinates** of a qubit (state) or ket-vector as a column vector:

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \alpha |0\rangle + \beta |1\rangle$$

with respect to the (orthonormal) **basis** $\{|0\rangle, |1\rangle\}$, i.e. the so-called standard base:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Representing a Qubit [∗]

A qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$ can be represented:

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\varphi}\sin(\theta/2)|1\rangle\,,$$

where $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$. Using polar coordinates we have:

$$|\psi\rangle = r_0 e^{i\phi_0}|0\rangle + r_1 e^{i\phi_1}|1\rangle\,,$$

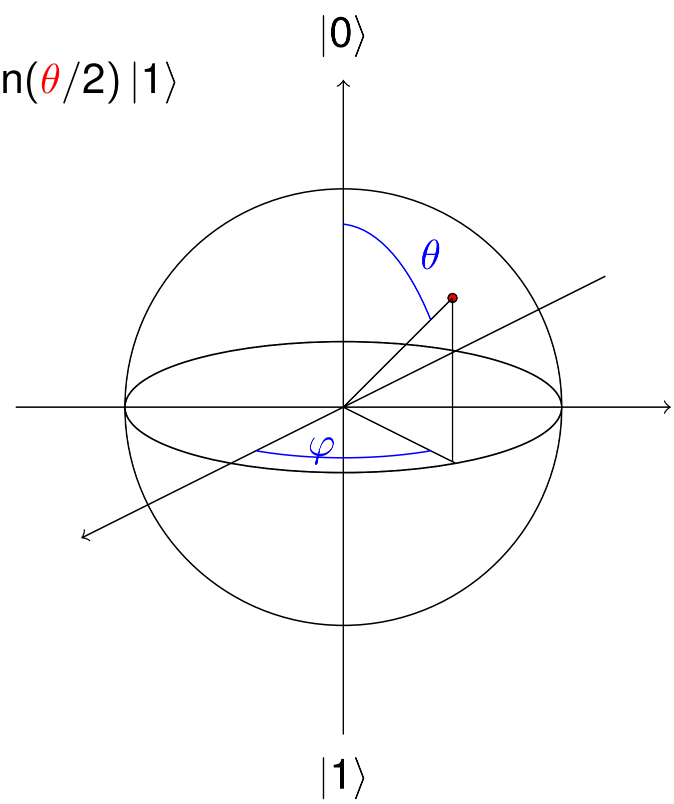
with $r_0^2 + r_1^2 = 1$. Take $r_0 = \cos(\rho)$ and $r_1 = \sin(\rho)$ for some ρ. Set $\theta/2 = \rho$, then $|\psi\rangle = \cos(\theta/2)e^{i\phi_0}|0\rangle + \sin(\theta/2)e^{i\phi_1}|1\rangle$, with $0 \leq \theta \leq \pi$, or equivalently

$$|\psi\rangle = e^{i\gamma}(\cos(\theta/2)|0\rangle + e^{i\varphi}\sin(\theta/2)|1\rangle),$$

with $\varphi = \phi_1 - \phi_0$ and $\gamma = \phi_0$, with $0 \leq \varphi \leq 2\pi$. The global **phase shift** $e^{i\gamma}$ is physically irrelevant (unobservable).

Bloch Sphere [∗]

$\cos(\theta/2)|0\rangle + e^{i\varphi}\sin(\theta/2)|1\rangle$



Change of Basis

We can represent (the coordinates of) any vector in $\mathbb{C}^n$ with respect to any basis we like.

For example, we can consider for qubits in $\mathbb{C}^2$ the (alternative) orthonormal basis:

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

and thus, vice versa:

$$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \quad |1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$$

A qubit is therefore represented in the two bases as:

$$\begin{aligned} \alpha\,|0\rangle + \beta\,|1\rangle &= \frac{\alpha}{\sqrt{2}}(|+\rangle + |-\rangle)\frac{\beta}{\sqrt{2}}(|+\rangle - |-\rangle) \\ &= \frac{\alpha+\beta}{\sqrt{2}}\,|+\rangle + \frac{\alpha-\beta}{\sqrt{2}}\,|-\rangle \end{aligned}$$

Linear Operators

Arguably, the best understood and controlled type of functions or maps between two vector spaces $\mathcal{V}$ and $\mathcal{W}$ are those preseving their basic algebraic structure.

Definition

A map $\mathbf{T} : \mathcal{V} \to \mathcal{W}$ between two vector spaces $\mathcal{V}$ and $\mathcal{W}$ is called a linear map if

1. $\mathbf{T}(\mathbf{x} + \mathbf{y}) = \mathbf{T}(\mathbf{x}) + \mathbf{T}(\mathbf{y})$ and
2. $\mathbf{T}(\alpha\mathbf{x}) = \alpha\mathbf{T}(\mathbf{x})$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ and all $\alpha \in \mathbb{K}$ (e.g. $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$).

For $\mathcal{V} = \mathcal{W}$ we talk about a (linear) operator on $\mathcal{V}$.

Images of the Basis

Like vectors, we can represent a linear operator **T** via its "coordinates" as a **matrix**. Again these depend on the **particular basis** we use.

Specifying the image of the base vectors determines – by **linearity** – the operator (or in general a linear map) uniquely.

Suppose we know the images of the basis vectors $|0\rangle$ and $|1\rangle$

$$\begin{aligned}\mathbf{T}(|0\rangle) &= T_{00}\,|0\rangle + T_{01}\,|1\rangle \\ \mathbf{T}(|1\rangle) &= T_{10}\,|0\rangle + T_{11}\,|1\rangle\end{aligned}$$

then this is enough to know the T_{ij}'s to know what **T** is doing to all vectors (as they are representable as linear combinations of the basis vectors).

Matrices

Using a "mathematical" indexing (starting from 1 rather ten 0), using the first index to indicate a **row** position and second for a **column** position, we can identify **T** with a matrix:

$$\mathbf{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = (T_{ij})_{i,j=1}^{n} = (T_{ij})$$

The application of **T** to a general vector (qubit) then becomes a simple matrix (pre-)multiplication:

$$\mathbf{T}\left(\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right) = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} T_{11}\alpha + T_{12}\beta \\ T_{21}\alpha + T_{22}\beta \end{pmatrix}$$

One can also express this, for $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ also as:

$$\mathbf{T}(|\psi\rangle) = \mathbf{T}(\alpha\,|0\rangle + \beta\,|1\rangle) = \alpha\mathbf{T}(|0\rangle) + \beta\mathbf{T}(|1\rangle) = \mathbf{T}\,|\psi\rangle$$

Matrix Multiplications

The application of a linear opertor **T** (represented by a matrix) to a vector **x** (represented via its coordinates) becomes:

$$\mathbf{T}(\mathbf{x}) = \mathbf{T}\mathbf{x} = (T_{ij})(x_i) = \sum_i T_{ij}x_i$$

The standard convention is pre-multiplication so as the sequence is the same as with application.

The composition of linear opertators **T** and **S** becomes also a matrix/matrix pre-multiplications:

$$\mathbf{T} \circ \mathbf{S} = \mathbf{TS} = (T_{ij})(S_{ki}) = \sum_i T_{ij}S_{ki}$$

Some authors use the more “computational” pre-multiplication.

Finite-dimensional linear operators (matrices) form a vector space and with the multiplication a (linear) **algebra**. Adding the adjoint operation (see below) turns this into a **C*****-algebra**.

Transformations

We can define a linear map **B** which implements the base change $\{|0\rangle, |1\rangle\}$ and $\{|+\rangle, |-\rangle\}$:

$$\mathbf{B} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Transforming the coordinates (x_i) with respect to $\{|0\rangle, |1\rangle\}$ into coordinates (y_i) using $\{|+\rangle, |-\rangle\}$ can be obtained by:

$$\mathbf{B}(x_i)_i = (y_i)_i \text{ and } \mathbf{B}^{-1}(y_i)_i = (x_i)_i$$

The matrix representation **T** of an operator using $\{|0\rangle, |1\rangle\}$ can be transformed into the representation **S** in $\{|+\rangle, |-\rangle\}$ via:

$$\mathbf{S} = \mathbf{BTB}^{-1}$$

Problem: It is not easy to compute inverse $\mathbf{B}^{-1}$, defined on implicitly by $\mathbf{BB}^{-1} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$ the identity (existence?!).

Adjoint Operator

For a matrix $\mathbf{T} = (T_{ij})$ its transpose matrix $\mathbf{T}^T$ is defined as

$$\mathbf{T}^T = (T_{ij}^T) = (T_{ji})$$

the conjugate matrix $\mathbf{T}^*$ is defined by

$$\mathbf{T}^* = (T_{ij}^*) = (T_{ij})^* = \overline{(T_{ji})}$$

and the adjoint matrix $\mathbf{T}^\dagger$ is given via

$$\mathbf{T}^\dagger = (T_{ij}^\dagger) = (T_{ji}^*) \quad \text{or} \quad \mathbf{T}^\dagger = (\mathbf{T}^*)^T = (\mathbf{T}^T)^*$$

Note that $(\mathbf{TS})^T = \mathbf{S}^T\mathbf{T}^T$ and thus $(\mathbf{TS})^\dagger = \mathbf{S}^\dagger\mathbf{T}^\dagger$.

In **mathematics** the adjoint operator is usually denoted by $\mathbf{T}^*$ (cf. conjugate in physics) and defined implicitly via:

$$\langle \mathbf{T}(\mathbf{x}), \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{T}^*(\mathbf{y}) \rangle \quad \text{or} \quad \langle \mathbf{T}^\dagger \mathbf{x} \mid \mathbf{y} \rangle = \langle \mathbf{x} \mid \mathbf{T}\mathbf{y} \rangle$$

Adjoint Vectors

Bra and ket vectors are also related using the adjoint:

$$|x\rangle^\dagger = \langle x|$$

or using their coordinates:

$$(x_i)^\dagger = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}^\dagger = \begin{pmatrix} \bar{x}_1 & \cdots & \bar{x}_n \end{pmatrix} = (\bar{x}^i)$$

The adjoint operator specifies the effect on dual vectors:

$$(\mathbf{T}\,|x\rangle)^\dagger = |x\rangle^\dagger\,\mathbf{T}^\dagger = \langle x|\,\mathbf{T}^\dagger$$

Unitary Operators

A square matrix/operator $\mathbf{U}$ is called unitary if

$$\mathbf{U}^\dagger \mathbf{U} = \mathbf{I} = \mathbf{U}\mathbf{U}^\dagger$$

That means $\mathbf{U}$'s inverse is $\mathbf{U}^\dagger = \mathbf{U}^{-1}$. It also implies that $\mathbf{U}$ is invertible and the inverse is easy to compute.

Quantum Mechanics requires that the dynamics or time evolution of a quantum state, e.g. qubit, is implemented via a unitary operator (as long as there is no measurement).

The unitary evolution of an (isolated) quantum state/system is a mathematical consequence of being a solution of the Schrödinger equation for some Hamiltonian operator $\mathbf{H}$.

Properties of Unitary Operators

Unitary operators generalise in some sense permutations (in fact every permutation of base vectors gives rise to a simple unitary map). They can also be seen as generalised rotations.

Unitary operators also preserve the "geometry" of a Hilbert space, i.e. they preserve the inner prduct:

$$\langle x | \, \mathbf{U}^\dagger \mathbf{U} \, | y \rangle = \langle x \mid y \rangle \, .$$

Any single qubit operation, i.e. unitary 2×2 matrix $\mathbf{U}$ can be expressed as via 4 (real) parameters:

$$\mathbf{U} = \begin{pmatrix} e^{i(\alpha - \beta/2 - \delta/2)} \cos \gamma/2 & e^{i(\alpha + \beta/2 - \delta/2)} \sin \gamma/2 \\ -e^{i(\alpha - \beta/2 + \delta/2)} \sin \gamma/2 & e^{i(\alpha + \beta/2 + \delta/2)} \cos \gamma/2 \end{pmatrix}$$

where α, β, δ and γ are real numbers.

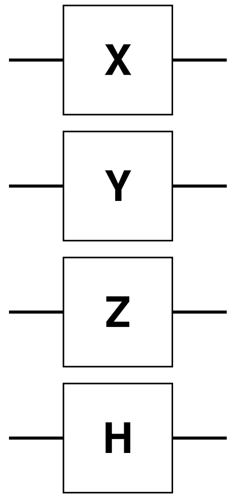
Basic 1-Qubit Operators

Pauli X-Gate $\mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

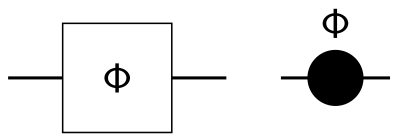
Pauli Y-Gate $\mathbf{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Pauli Z-Gate $\mathbf{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Hadamard Gate $\mathbf{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$



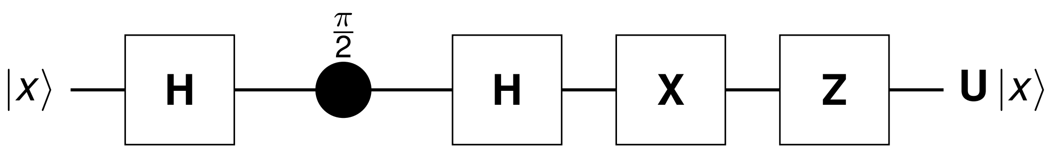
Phase Gate $\Phi = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$



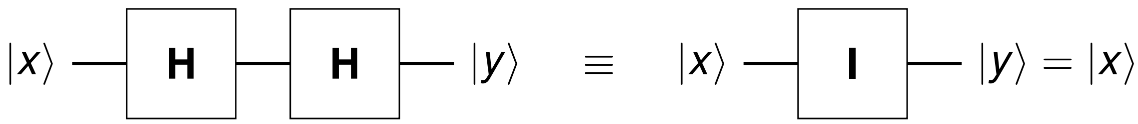
The Pauli-X gate is often referred to as NOT gate. Note that the notation for Hamiltonian and Hadamard gate are both **H**.

Graphical “Notation”

The product (combination) of unitary operators results in a unitary operator, i.e. with $\mathbf{U}_1, \ldots, \mathbf{U}_n$ unitary, the product $\mathbf{U} = \mathbf{U}_n \ldots \mathbf{U}_1$ is also unitary (Note: $(\mathbf{TS})^\dagger = \mathbf{S}^\dagger \mathbf{T}^\dagger$).



A simple example: $|y\rangle = \mathbf{HH}\,|x\rangle$ or $(|x\rangle\,;\ \mathbf{H};\ \mathbf{H} = |y\rangle)$:



because $\mathbf{H}^2 = \mathbf{I}$, i.e.

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$