Quantum Computation (CO484)

Quantum Algorithms: Deutsch Problem

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Balanced Functions

Definition

A function $f : \{0,1\} \rightarrow \{0,1\}$ is called balanced if $f(0) \neq f(1)$.

A balanced function on $\{0,1\}$ is one-to-one. There are two **balanced** functions on $\{0,1\}$:

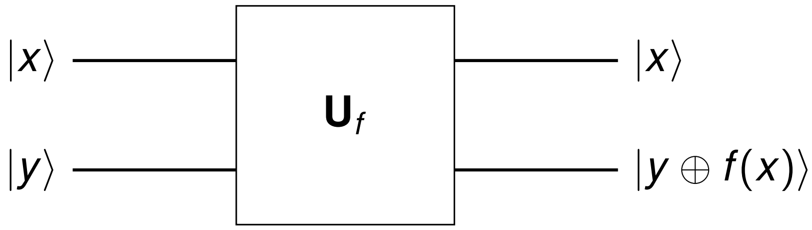
$$\begin{array}{ccc} 0 & \mapsto & 0 \\ 1 & \mapsto & 1 \end{array} \qquad \begin{array}{ccc} 0 & \mapsto & 1 \\ 1 & \mapsto & 0 \end{array}$$

and two **constant** functions on $\{0,1\}$.

$$\begin{array}{ccc} 0 & \mapsto & 0 \\ 1 & \mapsto & 0 \end{array} \qquad \begin{array}{ccc} 0 & \mapsto & 1 \\ 1 & \mapsto & 1 \end{array}$$

Deutsch Problem

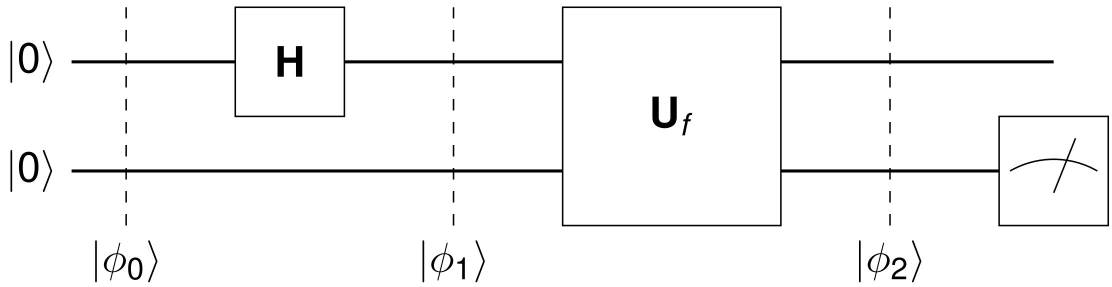
We can represent any function f on $\{0, 1\}$ by a quantum device i.e. as a unitary $\mathbf{U}_f$ on $\mathbb{C}^2 \otimes \mathbb{C}^2$ (making it thereby reversible):



$$\text{e.g.}\quad f = \begin{cases} 0 & \mapsto & 1 \\ 1 & \mapsto & 0 \end{cases} \quad \text{has}\quad \mathbf{U}_f = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Can one determine whether a function on $\{0, 1\}$ is balanced or not using $\mathbf{U}_f$ only once? Classically: Need to evaluate f twice.

A QCircuit for Deutsch – Version 1



$$|\phi_0\rangle = |0\rangle \otimes |0\rangle = |0\rangle\,|0\rangle = |00\rangle$$

$$|\phi_1\rangle = (\mathbf{H} \otimes \mathbf{I})\,|0\rangle\,|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\,|0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

$$|\phi_2\rangle = \frac{1}{\sqrt{2}}(|0, f(0)\rangle + |1, f(1)\rangle)$$

Superposition via Hadamard Gate

The important step in this circuit is involving the Hadamard Gate **H**. It's aim is to create a **superposition** of the base vectors $|0\rangle$ and $|1\rangle$, i.e. of all possible inputs:

$$
\begin{aligned}
&(\mathbf{H} \otimes \mathbf{I})(|0\rangle \otimes |0\rangle) = \\
&= \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) (|0\rangle \otimes |0\rangle) = \\
&= \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \otimes |0\rangle
\end{aligned}
$$

Swap Function – QCircuit Version 1

If we consider concretely the "swap" function $0 \mapsto 1$ and $1 \mapsto 0$.

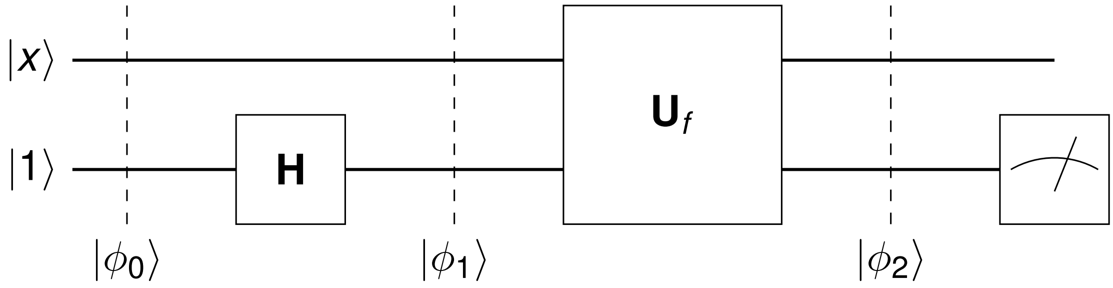
$$|\phi_0\rangle = (1, 0, 0, 0)^T$$

$$|\phi_1\rangle = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0)^T$$

$$|\phi_2\rangle = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

Problem: Measuring (either the first or second qubit) of $(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)^T$ in the standard base has a 50:50 chance to measure $|0\rangle$ and $|1\rangle$.

A QCircuit for Deutsch – Version 2



$$|\phi_0\rangle = |x\rangle \otimes |1\rangle = |x\rangle\,|1\rangle = |x,1\rangle$$

$$|\phi_1\rangle = (\mathbf{I} \otimes \mathbf{H})\,|x\rangle\,|1\rangle = \frac{1}{\sqrt{2}}\,|x\rangle\,(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}(|x,0\rangle - |x,1\rangle)$$

$$|\phi_2\rangle = |x\rangle\,(\frac{1}{\sqrt{2}}(|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle))$$

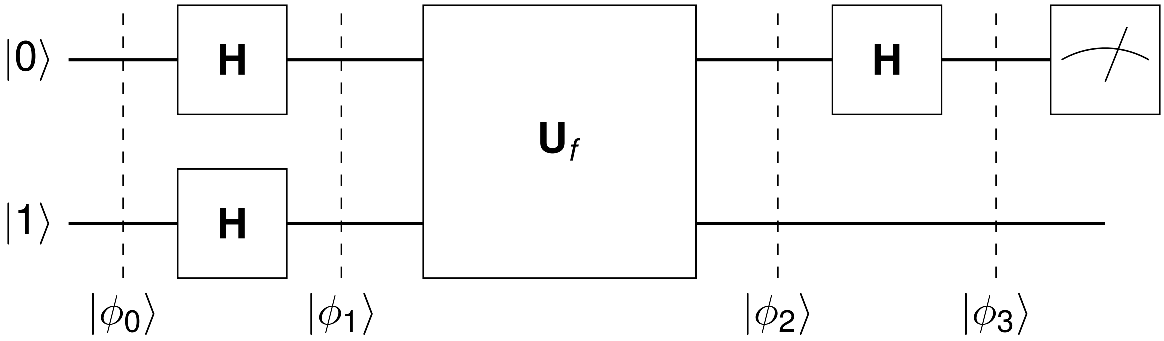
Final State – Version 2

By considering the function $\overline{f(x)}$ denoting the **opposite** of $f(x)$, that is: $\overline{f(x)} = (f(x) - 1) \bmod 2$ we get:

$$\begin{aligned}
|\phi_2\rangle &= |x\rangle\,\frac{1}{\sqrt{2}}(|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle) \\
&= \frac{1}{\sqrt{2}}\,|x\rangle\,(|f(x)\rangle - |\overline{f(x)}\rangle) \\
&= \begin{cases} \frac{1}{\sqrt{2}}\,|x\rangle\,(|0\rangle - |1\rangle) & \text{if } f(x) = 0 \\ \frac{1}{\sqrt{2}}\,|x\rangle\,(|1\rangle - |0\rangle) & \text{if } f(x) = 1 \end{cases} \\
&= (-1)^{f(x)}\frac{1}{\sqrt{2}}\,|x\rangle\,(|0\rangle - |1\rangle)
\end{aligned}$$

Problem: Measuring $|\phi_2\rangle$ does not reveal enough information.

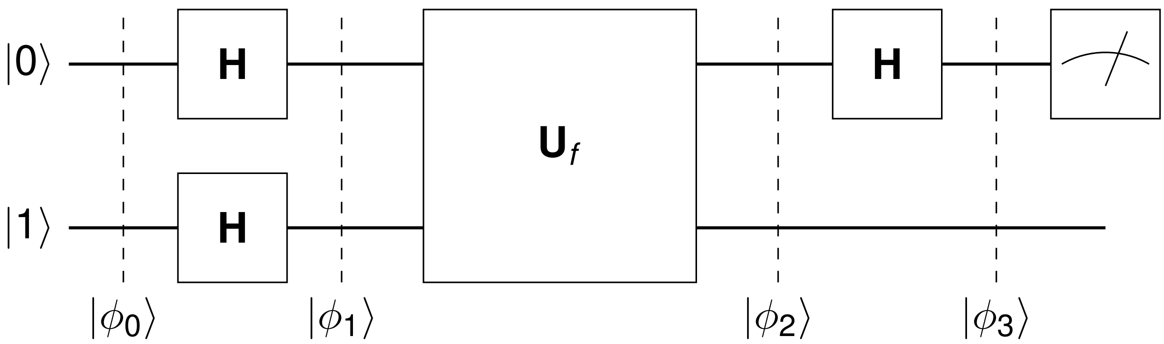
Deutsch's QCircuit



$$|\phi_0\rangle = |0\rangle \otimes |1\rangle = |0\rangle\,|1\rangle = |01\rangle$$

$$\begin{aligned} |\phi_1\rangle &= (\mathbf{H} \otimes \mathbf{H})(|0\rangle \otimes |1\rangle) \\ &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \\ &= \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle) \end{aligned}$$

Deutsch's QCircuit



$$|\phi_2\rangle = \left(\frac{1}{\sqrt{2}}((-1)^{f(0)}\,|0\rangle + (-1)^{f(1)}\,|1\rangle)\right) \otimes \left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)$$

For example, for the "swap" function, we get for the **top** qubit:

$$|\phi_2\rangle_1 = \frac{1}{\sqrt{2}}((-1)\,|0\rangle + (+1)\,|1\rangle) = -\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

The Final State

Investigating $(-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle$ closely, we describe $|\phi_2\rangle$ as:

$$|\phi_2\rangle = \begin{cases} (\pm 1)(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)) \otimes (\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)) & \text{if } f \text{ constant} \\ (\pm 1)(\frac{1}{\sqrt{2}}(|1\rangle - |0\rangle)) \otimes (\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)) & \text{if } f \text{ balanced} \end{cases}$$

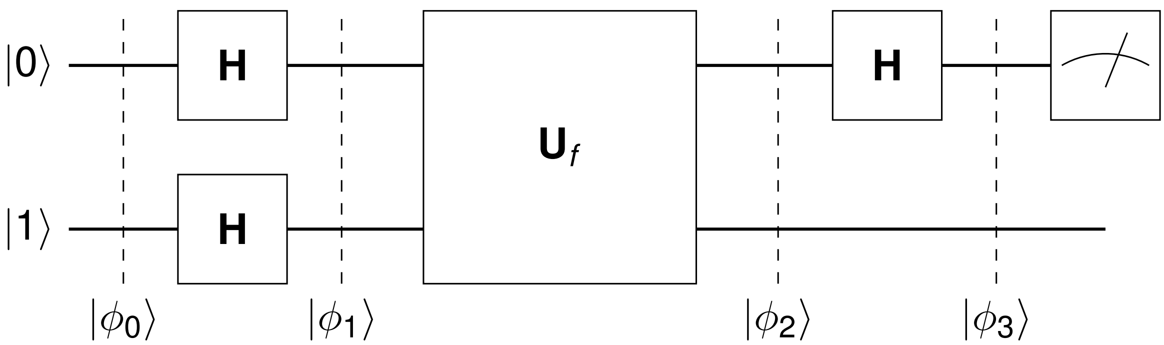
Applying Hadamard to the first qubit gives:

$$|\phi_3\rangle = \begin{cases} (\pm 1)\,|0\rangle \otimes (\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)) & \text{if } f \text{ constant} \\ (\pm 1)\,|1\rangle \otimes (\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)) & \text{if } f \text{ balanced} \end{cases}$$

For example, for the “swap” function, we get:

$$|\phi_3\rangle = (-1)\,|1\rangle \otimes (\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)).$$

The Solution to Deutsch’s Problem



$$|\phi_3\rangle = \begin{cases} (\pm 1)\,|0\rangle \otimes (\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)) & \text{if } f \text{ constant} \\ (\pm 1)\,|1\rangle \otimes (\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)) & \text{if } f \text{ balanced} \end{cases}$$

Measuring the first/top qubit (in the standard base) now indicates with probability 1 whether we are in state $|0\rangle$ or $|1\rangle$.
If we measure/observe

$|0\rangle$ then f is a constant function,
$|1\rangle$ then f is a balanced function.