Feasibly Constructive Proof of Schwartz-Zippel Lemma and the Complexity of Finding Hitting Sets

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Abstract

The Schwartz-Zippel Lemma states that if a low-degree multivariate polynomial with coefficients in a field is not zero everywhere in the field, then it has few roots on every finite subcube of the field. This fundamental fact about multivariate polynomials has found many applications in algorithms, complexity theory, coding theory, and combinatorics. We give a new proof of the lemma that offers some advantages over the standard proof.

First, the new proof is more constructive than previously known proofs. For every given side-length of the cube, the proof constructs a polynomial-time computable and polynomial-time invertible surjection onto the set of roots in the cube. The domain of the surjection is tight, thus showing that the set of roots on the cube can be compressed. Second, the new proof can be formalised in Buss' bounded arithmetic theory S^1_2 for polynomial-time reasoning. One consequence of this is that the theory S^1_2+$\mathsf{dWPHP}(\mathsf{PV})$ for approximate counting can prove that the problem of verifying polynomial identities (PIT) can be solved by polynomial-size circuits. The same theory can also prove the existence of small hitting sets for any explicitly described class of polynomials of polynomial degree.

To complete the picture we show that the existence of such hitting sets is *equivalent* to the surjective weak pigeonhole principle $\mathsf{dWPHP}(\mathsf{PV})$, over the theory S^1_2. This is a contribution to a line of research studying the reverse mathematics of computational complexity (cf. Chen-Li-Oliveira, FOCS'24). One consequence of this is that the problem of constructing small hitting sets for such classes is complete for the class APEPP of explicit construction problems whose totality follows from the probabilistic method (Kleinberg-Korten-Mitropolsky-Papadimitriou, ITCS'21; cf. Korten, FOCS'21). This class is also known and studied as the class of Range Avoidance Problems (Ren-Santhanam-Wang, FOCS'22).

Contents

1 Introduction

We study constructive proofs of the well-known Schwartz-Zippel Lemma, and their applications in complexity theory. This statement is sometimes referred to as the "Schwartz-Zippel-DeMillo-Lipton Lemma" following [47, 42, 15]. However, this result goes back to Øystein Ore, 1922 [34] (over finite fields) and was subsequently rediscovered by other authors; see [3] who unearthed the history of this lemma. Due to this complicated history, it is sometimes called "The Polynomial Identity Lemma".

Theorem 1.1 (Schwartz-Zippel Lemma)**.** *Let $\mathbb{F}$ be a field, let $\overline{x}$ be a set of n indeterminates, let $S \subseteq \mathbb{F}$ be a finite subset of field elements, and let $P(\overline{x})$ be a multivariate polynomial in the indeterminates $\overline{x}$ with coefficients in the field $\mathbb{F}$ and maximum individual degree at most d. Then, either every point in $\mathbb{F}^n$ is a root of $P(\overline{x})$, or the number of roots in S^n is at most $d{\cdot}n{\cdot}|S|^{n-1}$. In particular, the number of roots in S^n is either $|S|^n$ or at most $d{\cdot}n{\cdot}|S|^{n-1}$.*[1]

The Schwartz-Zippel Lemma is a cornerstone in randomised algorithms and the use of randomness in computing, with a wide range of applications in computational complexity, coding theory, graph algorithms, and algebraic computation. The lemma shows that evaluating a non-zero polynomial on inputs chosen randomly from a large enough set is likely to find an input that produces a non-zero result. This offers a fast test with good guarantees for checking if a polynomial is identically zero.

While the lemma provides significant efficiency advantages in its applications, it is based on an existential statement—the existence of many non-roots for non-zero polynomials—without offering a deterministic way to find these non-roots or easily witness their absence. Accordingly, the standard textbook proof of the lemma, which goes by induction on the number of variables, although simple, does not reveal a feasible constructive argument. Specifically, it treats multivariate polynomials as potentially exponential sums of monomials. This non-constructive nature of the lemma is one reason why providing feasible constructive proofs has been challenging (cf. [28]).

Although different in nature, feasible constructivity in proofs and algorithms often go hand in hand, either informally or, at times, formally through translation theorems between proofs and computation. This work presents a new proof of the Schwartz-Zippel Lemma that fits within the framework of feasibly constructive proofs in a precise manner. We then demonstrate several applications of this proof, both in feasible constructive mathematics and in computational complexity, as we explain in the following.

Organisation of the introduction. In Section 1.1 we discuss the motivation behind seeking feasibly constructive proofs in general, and specifically the approach taken here to carry

[1] Our version of the lemma is for *maximum individual degree* and is closely related to Zippel's version of the lemma in [47]. Zippel's bound, also for maximum individual degree, is slightly tighter than the one stated here, namely: $|S|^n \cdot (1 - (1 - d/|S|)^n) \leq d \cdot n \cdot |S|^{n-1}$. There is also a better known version of the lemma for *total degree* D, where the bound is $D \cdot |S|^{n-1}$. Note that $d \leq D \leq d \cdot n$, so the bound for total degree is never further than a factor of n from our bound. In the regime where $|S|$ is bigger than n and d, which is the setting of most applications, all bounds are equally useful and yield similar conclusions. See [30] and the discussion there, for a comparison of the bounds, and for a discussion on how tight they are.

out the new proof within the formal logic framework of bounded arithmetic, which are formal theories accompanying and complementing the development of complexity theory. In Section 1.2 we describe (in the "meta-theory") the new feasibly constructive proof of Schwartz-Zippel. For those interested to first see the new proof, it is presented at the end of Section 1.2. We precede it with an exposition that aims to explain the intuition behind the new proof. In Section 1.3 we discuss applications of the new proof to bounded arithmetic, where our new argument helps to settle the open problem of formalising the Schwartz-Zippel Lemma in bounded arithmetic. Specifically, in Section 1.3.2 we discuss how to prove the existence of small hitting sets and hence formalise Polynomial Identity Testing (PIT) in the theory. In Section 1.3.3 we describe a "reversal" theorem, in which the dual weak pigeonhole principle—namely, the statement that a function cannot map surjectively a small domain onto a large range—is shown to be equivalent to the existence of small hitting sets. And in Section 1.3.4 we show that this reversal theorem implies that finding hitting sets is complete for the class of Range Avoidance Problems (aka APEPP).

1.1 The Quest for Feasibly Constructive Proofs

While existence proofs, particularly those establishing the presence of certain combinatorial objects, are very useful—for instance, in probabilistic combinatorics, where one seeks to identify objects with certain properties that are as large or small as possible, such as expanders or combinatorial designs—it is widely acknowledged that constructive arguments, which provide explicit methods for constructing these objects, are often even more fruitful. This is especially significant in algorithmic contexts, where the utility of the objects (e.g., expanders) depends on their "explicitness", meaning that they must be feasibly computable, typically in polynomial time.

In computational complexity theory, the notions of constructivity and explicitness are equally critical. Fundamental questions about separating complexity classes hinge on explicit languages, such as the satisfiability problem (SAT), and whether SAT can be solved by polynomial-size circuits. For random, non-explicit languages, the analogous problems are almost trivially resolved: a simple counting argument shows that there exist non-explicit Boolean functions that cannot be computed by Boolean circuits of polynomial-size.

A related but distinct approach to feasible constructivity in complexity theory is found in the framework of *bounded arithmetic*, which is the field that studies the computational complexity of concepts needed to prove different statements. In this setting, one aims to formalise constructivity in a more rigorous and systematic manner. Bounded arithmetic is a general name for a family of weak formal theories of arithmetic (that is, natural numbers, and whose intended model is $\mathbb{N}$). These theories are characterized by their axioms and language (set of symbols), starting from a basic set of axioms providing the elementary properties of numbers. Each bounded arithmetic theory possesses different additional axioms postulating the existence of different sets of numbers, or different kinds of induction principles. Based on its specific axioms, each theory of bounded arithmetic proves the totality of functions from different complexity classes (e.g., polynomial-time functions). We can typically consider such theories as working over a logical language that contains the function symbols of that

prescribed complexity class. In this sense, proofs in the theory use concepts from a specific complexity class, and we can say that the theory captures "reasoning in this class" (e.g., "polynomial-time reasoning").

In the current work we shall start with a standard naturally written constructive proof (in the "meta-theory") of the Schwartz-Zippel Lemma (Section 1.2), following the first, less formal approach, to constructivity. We then show how the new proof fits in the formal approach to feasible constructivity of bounded arithmetic. We moreover exemplify the usual benefits of such proofs by showing applications both in bounded arithmetic and computational complexity.

Background on theories of bounded arithmetic, their utility and applications. While the first theory for polynomial-time reasoning was the equational theory PV considered by Cook [14], bounded arithmetic goes back to the work of Parikh [35] and Paris-Wilkie [36]. In a seminal work, Buss [8] introduced other theories of bounded arithmetic and laid much of the foundation for future work in the field.

Using formal theories of bounded arithmetic is important for several reasons. First, it provides a rigorous framework to ask questions about provability, independence, and the limits of what can be proved by which means and arguments. The quest for "barriers" in computational complexity—namely, the idea that some forms of arguments are futile as a way to solve the fundamental hardness problems in complexity—such as Relativisation by Baker, Gill and Solovay [6], Natural Proofs by Razborov and Rudich [40] or Algebrisation by Aaronson and Wigderson [1], has played an important role in complexity theory. Nevertheless, the language of formal logic provides a more systematic framework for such a project (cf. [2, 19] and the recent work [11]). In that sense, bounded arithmetic allows us to identify suitable logical theories capable of formalising most contemporary complexity theory, and determine whether the major fundamental open problems in the field about lower bounds are provable or unprovable within these theories.

Furthermore, bounded arithmetic serves as a framework in which the *bounded reverse mathematics* program is developed (in an analogy to Friedman and Simpson reverse mathematics program [44]). In this program, one seeks to find the weakest theory capable of proving a given theorem. In other words, we seek to discover what are the axioms that are not only sufficient to prove a certain theorem, but rather are also *necessary*. Special theorems of interest are those of computer science and computational complexity theory. The motivation is to shed light on the role of complexity classes in proofs, in the hope to delineate, for example, those concepts that are needed to progress on major problems in computational complexity from those that are not. For instance, it has been identified that apparently most results in contemporary computational complexity can be proved using polynomial-time concepts (e.g., in PV) (cf. [38]), and it is important to understand whether stronger theories and concepts are needed to prove certain results.

Accordingly, recent results in bounded arithmetic [11] seek to systematically build a bounded reverse mathematics of complexity lower bounds, in particular, as these are perhaps the most fundamental questions in complexity. This serves to establish complexity lower bounds as "fundamental mathematical axioms" which are equivalent, over the base theory,

to rudimentary combinatorial principles such as the pigeonhole principle or related principles. Indeed, we will show that the (upper bound) statement about the existence of small hitting sets is equivalent to the (dual weak) pigeonhole principle. (It is interesting to note that the existence of explicit small hitting sets, which implies efficient PIT, is also a disguised lower bound statement as shown by Kabanets and Impagliazzo [23]; though we do not attempt to formalise or pursue this direction in the current work.)

Another advantage of bounded arithmetic comes in the form of "witnessing" theorems. These are results that automatically convert formal proofs (of low logical-complexity theorems, namely, few quantifier alternations) in bounded arithmetic to feasible algorithms (usually, deterministic polynomial-time ones). Witnessing theorems come in different flavours and strength, and recent work show the advantage of this method in both lower and upper bounds [10, 18, 29].

Moreover, the somewhat forbidding framework of bounded arithmetic forces one to think algorithmically from the get go, optimising constructions. This resulted in new arguments to existing results, which proved very useful in complexity, beyond the scope of bounded arithmetic. A celebrated example is Razborov's new coding argument of the Håstad's switching lemmas [17], which emerged as work in bounded arithmetic [39]. (Intriguingly, our new argument for Schwartz-Zippel will also be based on a coding argument.)

Randomness and feasibly constructive proofs. One central part of complexity that was challenging to fit into bounded arithmetic is that of random algorithms. Randomness usually entails thinking of probability spaces of exponential size (e.g., the outcome of n coin flips), and so cannot be directly used in most bounded arithmetic theories which cannot state the existence of exponential size sets. Initial work by Wilkie (unpublished) [26, Theorem 7.3.7], taken as well by Thapen [45], and systematically developed in a series of works of Jeřábek (cf. [21]), concerned how to work with randomness in bounded arithmetic. Nevertheless, one of the most well-known examples for using randomness in computing, the question of formulating Schwartz-Zippel and polynomial identity testing in particular, was left open due to the exponential nature of the standard argument (i.e., the need to treat polynomials as sums of exponential many monomials; see the first paragraph in Section 1.2). For example, Lê and Cook in [28] list this as an open question (where they settled for formalising a special case of the Schwartz-Zippel Lemma).

The formalisation of Schwartz-Zippel Lemma and PIT in bounded arithmetic and its consequences we present, hopefully exemplify several of the benefits of bounded arithmetic described above. It serves to fill in the missing link in the formalisation of complexity of randomness; it produces a new (coding) argument of Schwartz-Zippel lemma that may be of independent interest; it establishes the existence of hitting sets as equivalent to the dual weak pigeonhole principle, and thus provides further examples of the "axiomatic" nature of building blocks in complexity theory; lastly, it has consequences to computational complexity, by showing that hitting sets are complete for the class of range avoidance problems.

1.2 New Proof of the Schwartz-Zippel Lemma

The standard Schwartz-Zippel proof proceeds by induction on the number of variables n but is not a feasibly constructive proof. It is non-constructive in the sense that each inductive step involves stating properties of objects that are of exponential size. For example, in step i of the induction, the proof states that the current polynomial on $n-i+1$ variables can be rewritten into a univariate polynomial in the last variable, with coefficients taken from the ring of polynomials in the first $n-i$ variables. The non-constructive character of the proof stems then from the fact that there is no (known) efficient way of iterating this rewriting n times, unless the polynomial is given in explicit sum of monomials form. Note, however, that the sum of monomials form is typically of exponential size.

Moshkovitz [32] provided an alternative proof of the Schwartz-Zippel Lemma, but only for finite fields. The proof in [32] does not use explicit induction, but it is anyway unclear how to make it strictly constructive in our sense, namely how to identify polynomial-time algorithms for the concepts used in the proof, and then use these to formalise the whole argument in a relatively weak theory. We refer to Moshkovitz' proof again later in this section to compare it with our approach.

Towards our new proof. For the sake of exposition, let us begin with two natural but failed attempts to a new proof. In what follows, let $P(\overline{x})$ be a polynomial over the field $\mathbb{F}$, with n variables and maximum individual degree d, and let $\overline{a} \in \mathbb{F}^n$ be a given non-root; $P(\overline{a}) \neq 0$. Let $S \subseteq \mathbb{F}$ be a finite subset of the field. In the *first attempt*, we try to cover the cube S^n with at most $n \cdot |S|^{n-1}$ lines, each emanating from the non-root $\overline{a}$. These lines are the subsets of $\mathbb{F}^n$ of the form $\{\overline{a} + t \cdot \overline{b} : t \in \mathbb{F}\}$ for some $\overline{b} \in \mathbb{F}^n$. For each such line L, note that P retricted to L, defined as $P_L(t) := P(\overline{a} + t \cdot \overline{b})$, is a non-zero *univariate* polynomial of degree at most d. It is non-zero because it evaluates to $P(\overline{a}) \neq 0$ at $t = 0$, and it has degree at most d because it is a linear restriction of P. Therefore, by the fundamental theorem for (univariate) polynomials, each such line has at most d roots, and since the lines cover S^n, we count at most $d \cdot n \cdot |S|^{n-1}$ roots in S^n in total, and we are done.

The problem with this approach is that it is not always possible to cover $|S|^n$ with at most $n \cdot |S|^{n-1}$ lines emanating from a single point $\overline{a}$. A simple counterexample can be found already at dimension $n = 2$ with $S = \{0, 1, \ldots, q-1\}$ and $\overline{a} = (0, 0)$: the $2q-1$ points of the form $(i, q-1)$ or $(q-1, i)$ with $i \in S$ require each its own line emanating from the origin, and the remaining $q^2 - 2q - 2$ points cannot be covered with one more line.

In the *second attempt*, we want to cover the cube S^n again with at most $n \cdot |S|^{n-1}$ lines, but now we try with *parallel lines*. For example we could consider the set of axis-parallel lines of the form $\{(c_1, \ldots, c_{k-1}, t, c_{k+1}, \ldots, c_n) : t \in \mathbb{F}\}$ with $c_j \in S$ for all $j \neq k$, for some fixed $k = 1, \ldots, n$. The problem with this approach now is that it is not clear that all such lines will go through some non-root of P. It is tempting to consider some sets of parallel lines that are more related to the given non-root $\overline{a}$, such as the set of lines of the form $\{\overline{b} + t \cdot \overline{a} : t \in \mathbb{F}\}$ whose *gradient* is $\overline{a}$. This is indeed the approach taken by Moshkovitz in [32], but as far as we can see this does not work for arbitrary non-roots $\overline{a}$, and works only for a specific kind of non-roots that seem hard to find in the first place.

Our approach. We are now ready to explain the two new ideas that we use in our proof, and how they overcome the obstacles of the previous two attempts. First, instead of *covering* the roots in S^n with $n \cdot |S|^{n-1}$ lines where the polynomial is non-zero, we are going to *encode* each root in S^n with one in $n \cdot |S|^{n-1}$ such lines, together with an additional number i in $1, \ldots, d$. The lines we use to encode the roots are not necessarily pair-wise parallel, though each line will be parallel to one of the axes. Second, to actually find this line, our proof uses a *hybrid-type argument*. Concretely, to encode the root $\overline{c}$, we start at a line through $\overline{a}$ and end at a line through $\overline{c}$. Along the way, the process travels across at most n axis-parallel lines of the cube S^n, changing the dimension of travel at each step. The hybrid-argument is used to preserve the property that the restriction of P to the current line is still a non-zero polynomial. The exact details of how this is done are explained below.

New proof of Schwartz-Zippel lemma: Let $\overline{a} = (a_1, \ldots, a_n) \in \mathbb{F}^n$ be such that $P(\overline{a}) \neq 0$, and let S be a finite subset of the field $\mathbb{F}$. Each vector $\overline{c} = (c_1, \ldots, c_n) \in S^n$ of field elements in S can be encoded with n numbers, each from 1 to $|S|$, by identifying each c_i with its position in an arbitrary ordering of the finite set S. Our goal is to *encode the roots of P in S^n with shorter codewords*. To achieve this we will use only the fact that we know a non-root $\overline{a} \in \mathbb{F}^n$.

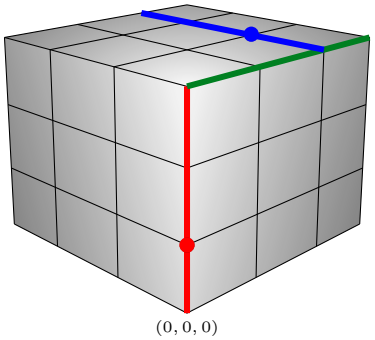
Let $\overline{c} = (c_1, \ldots, c_n) \in S^n$ be such that $P(\overline{c}) = 0$. Find the minimal index k between 1 and n such that $P(c_1, \ldots, c_{k-1}, a_k, a_{k+1}, \ldots, a_n) \neq 0$ while $P(c_1, \ldots, c_{k-1}, c_k, a_{k+1}, \ldots, a_n) = 0$. Such a k must exist since by assumption $P(\overline{a}) \neq 0$ and $P(\overline{c}) = 0$. Observe that, if we are given both $\overline{a}$ and $\overline{c}$, then finding k is easy by looping through the coordinates from left to right. The argument hinges on the following observation:

> *Knowing this k allows us to encode, given $c_1, \ldots, c_{k-1}, c_{k+1}, \ldots, c_n$, the field element $c_k \in S$ by a single number i between 1 and d.*

Therefore, we can use i, together with k and the positions of $c_1, \ldots, c_{k-1}, c_{k+1}, \ldots, c_n$ in the fixed ordering of S, as a code for $\overline{c}$. This shows that the set of roots in S^n can be encoded using only $d \cdot n \cdot |S|^{n-1}$ numbers instead of the trivial $|S|^n$ bound, concluding the argument.

In detail, consider the root $\overline{c} = (c_1, \ldots, c_{k-1}, c_k, c_{k+1} \ldots, c_n)$, where k is the minimal index as above. We encode $\overline{c}$ by the $n-1$ elements $c_1, \ldots, c_{k-1}, c_{k+1}, \ldots, c_n$ from S, together with k and a second index i such that $c_k \in S$ is the i-th root in S of the univariate polynomial $Q(t) = P(c_1, \ldots, c_{k-1}, t, a_{k+1}, \ldots, a_n)$. This encoding works because given the numbers k and i, and the elements $(c_1, \ldots, c_{k-1}, c_{k+1} \ldots, c_n)$ we can recover $\overline{c}$. Indeed, by the choice of k and i, the univariate polynomial $Q(t)$ is *non-zero and has degree at most d*, which means that it has no more than d roots in $\mathbb{F}$. By looping through S we can find its i-th root in S which, by construction, is c_k. The fact that each root of P in S^n is coded by $n-1$ of its components and the two positive integers k and i finishes the proof as it means that the total number of such roots is bounded above by the number of all possible such codes: $d \cdot n \cdot |S|^{n-1}$.

Illustration of the encoding process. Refer to the cube on the right. The x, y, z axes are represented by the vertical dimension, and the two horizontal dimensions, respectively. The given non-root is the red dot $\overline{a} = (1, 0, 0)$. The root to encode is the blue dot $\overline{c} = (3, 2, 1)$. The process starts at the red line $(t, 0, 0)$ through $\overline{a}$, parallel to the x axis. Replacing the first component a_1 by c_1, we check if $P(c_1, a_2, a_3) = 0$. If not, we travel along the red line for $c_1 - a_1 = 2$ units to reach the green line $(3, t, 0)$, parallel to the y axis. Next we check if $P(c_1, c_2, a_3) = 0$. If not, we travel along the green line for $c_2 - a_2 = 2$ units to reach the blue line $(3, 2, t)$, parallel to the z axis. We test now if $P(c_1, c_2, c_3) = 0$, which checks, because $\overline{c}$ is a root. The journey ends here. In this example it took us $k = 3$ steps to reach the root. Note that this was the first k in $1, \ldots, n$ that caused P to vanish on the *hybrid* $(c_1, \ldots, c_{k-1}, c_k, a_{k+1}, \ldots, a_n)$.



The blue line, call it L, is the one we use to encode the root $\overline{c}$. To encode it, we use the index of c_k as a root of the univariate polynomial $P_L(t)$ obtained by restricting P to this line. If this index is i in $1, \ldots, d$, then we encode the root $\overline{c}$ by $(i, k, (c_1, \ldots, c_{k-1}, c_{k+1}, \ldots, c_n))$. In the example, if the index i happens to be 1, then we use $(1, 3, (3, 2))$. Note that this encoding depends on the given non-root $\overline{a}$, but any non-root serves the purpose of encoding all roots in S^n.

1.3 Applications

1.3.1 Schwartz-Zippel Lemma in the Theory

We begin with a short informal description of the theories, language and axioms we shall use.

PV: This is the language with a function symbol for every polynomial-time function, with its meaning specified by the equations that define it via Cobham's bounded recursion on notation.

S^1_2: The first level of Buss' family of theories [8] for basic number theory whose definable functions are precisely the polynomial-time functions. It contains basic axioms for properties of numbers (e.g., associativity of product), together with a polynomial induction axiom for NP-predicates. The extension of S^1_2 with all PV-symbols and the Cobham equations as axioms is denoted by $\mathsf{S}^1_2(\mathsf{PV})$. The theory has the same theorems as S^1_2 in the base language (see [8]), and it is customary to abuse notation and still call it S^1_2 instead of the heavier $\mathsf{S}^1_2(\mathsf{PV})$.

dWPHP(PV): The set of *dual weak pigeonhole principle* axioms $\mathsf{dWPHP}(f)$, for every polynomial-time function symbol $f \in \mathsf{PV}$. This axiom states the simple counting principle that a function f cannot map surjectively a domain of size N to a range of size $2N$ or more; namely, there is a point in the set of size $2N$ that is not covered by f. Wilkie

(unpublished; see [26, Theorem 7.3.7]) observed the connection between this principle and randomness in computation (within bounded arithmetic): roughly speaking, when f has a small domain but much larger co-domain, with high probability a point in the co-domain will not be covered by f. Hence, the ability to pick such a point is akin to witnessing this probabilistic argument.

S^1_2+dWPHP(PV): S^1_2 (indeed $\mathsf{S}^1_2(\mathsf{PV})$; see above), augmented with the axioms $\mathsf{dWPHP}(f)$ for every polynomial-time function symbol $f \in \mathsf{PV}$. This is a theory that can serve as a basis for probabilistic reasoning (close to Jeřábek's theory for approximate counting [21]; cf. [20]).

With this notation we can now state the form of Schwartz-Zippel Lemma that we prove. Here, and in the rest of this introduction, let $[q]$ denote the set $\{1, \ldots, q\}$.

Theorem 1.2 (Schwartz-Zippel Lemma in S^1_2; informal, see Theorem 3.9)**.** *Let P be a polynomial of degree d, given as an algebraic circuit, with integer coefficients and n variables. Then, either P is zero everywhere on $\mathbb{Z}$, or for every positive integer q there is a (polynomial-time) function f that given any non-root $\overline{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ with $P(\overline{a}) \neq 0$, returns a function $f(\overline{a}) : \mathsf{codes} \to \mathsf{roots}$ that maps the set of codes* surjectively onto *the set of roots in the cube $[q]^n$, and $|\mathsf{codes}| \leq d \cdot n \cdot q^{n-1}$.*

Here the codes and the function f are defined according to the encoding scheme in Section 1.2. Since given a non-root $\overline{a}$ the function $f(\overline{a})$ is (provably) surjective onto the roots in the cube $[q]^n$, the number of roots of P in the cube is at most the number of codes, or $|\mathsf{codes}| \leq d \cdot n \cdot q^{n-1}$. To actually reason in the theory about the size of exponential-size sets like codes we could have chosen to invoke approximate counting (based on Jeřábek's theories [21], which would require the inclusion of the dual weak pigeonhole principle dWPHP). However, we opt not to do this for the Schwartz-Zippel Lemma. Rather, we will show that this formulation of the Schwartz-Zippel Lemma, together with the $\mathsf{dWPHP}(\mathsf{PV})$ axiom, suffices to apply the lemma in its standard applications, such as PIT and finding small hitting sets (see below).

En route to the proof of Theorem 1.2, we prove in the theory S^1_2 one half of the *Fundamental Theorem of Algebra* (FTA). This is the fundamental theorem of univariate polynomials stating that every non-zero polynomial of degree d with complex coefficients has exactly d complex roots. The theorem naturally splits into two halves: the half that states that there are *at least d roots*, and the half that states that there are *at most d roots*. While the *at least* statement relies on special properties of the complex numbers, the proof of the *at most* statement relies only on the fact that univariate polynomials over a field admit Euclidean division. In particular, the *at most* statement holds also for polynomials over any subring of a field; e.g., the integers.

Theorem 1.3 (Half of Fundamental Theorem of Algebra in S^1_2 informal; see Lemma 3.4)**.** *Every non-zero univariate polynomial of degree d with integer coefficients has at most d roots on (every finite subset of) $\mathbb{Z}$.*

While the underlying idea of this proof is standard, it is somewhat delicate to carry out the argument in S^1_2 because we need to keep track of the bit-complexity of the coefficients that appear along the way in the computations. It is well-known that certain widely-used algorithms working with integers or rational numbers, including Gaussian Elimination, could incur exponential blow-ups in bit-complexity if careless choices were made in their implementation; cf. [16]. We also note that Jeřábek [22] formalised Gaussian Elimination over rationals in S^1_2, and proved also the same half of the FTA that we prove, but only for finite fields, where exponential blow ups cannot occur.

1.3.2 Existence of Hitting Sets and PIT in the Theory

For a field $\mathbb{F}$ and a set of algebraic circuits $\mathscr{C}$ over $\mathbb{F}$ with n variables, we say that a set $H \subseteq \mathbb{F}^n$ is a *hitting set for* $\mathscr{C}$ if for every non-zero polynomial P in $\mathscr{C}$ there exists a point $\overline{a} \in H$ such that $P(\overline{a}) \neq 0$. In other words, if P is non-zero, H 'hits' it. Hitting sets are important because when they are explicit and small they allow for derandomization of PIT: running through the full hitting set one can test if a given algebraic circuit is the zero polynomial or not.

By Theorem 1.2, the theory S^1_2 proves (by means of a surjective map) that every non-zero n-variable algebraic circuit with small degree d has relatively few roots in $[q]^n$. By a counting argument (or the union bound, cf. [43, Theorem 4.1]), it follows that for any given bounds d and 2^m on the degree and the number of circuits in the class $\mathscr{C}$, there is a set $H \subseteq [q]^n$ of $\mathsf{poly}(n, d, m)$ points, with $q = \mathsf{poly}(n, d)$, that intersects the set of non-roots of every non-zero circuit in the class. This set H is thus a hitting set of polynomial size, and we say it is a hitting set for $\mathscr{C}$ *over* $[q]$. We show that this counting argument can now be formalised in the theory $\mathsf{S}^1_2 + \mathsf{dWPHP}(\mathsf{PV})$.

Theorem 1.4 (Small Hitting Sets Exist in $\mathsf{S}^1_2 + \mathsf{dWPHP}(\mathsf{PV})$; informal, see Theorem 4.4)**.** *For every class $\mathscr{C}$ of algebraic circuits with integer coefficients that is definable in the theory, with n variables, polynomial degree, and polynomial size, there exists a polynomial-size hitting set for $\mathscr{C}$ over $[q]$ with $q = \mathsf{poly}(n)$.*

The argument in the theory makes two uses of the axiom dWPHP and is roughly as follows. We begin by showing that, if q is sufficiently large but polynomial, then a non-zero polynomial with n variables and polynomial degree always has non-root $\overline{a}$ in $[q]^n$. To see this, recall the function $f(\overline{a}) : \mathsf{codes} \to \mathsf{roots}$ from Theorem 1.2, that given a polynomial P and a non-root $\overline{a}$ of P, surjectively maps all codes of roots to the roots of P. Note that the set roots is a subset of $[q]^n$, which has size q^n, and recall that the set codes has size at most $d \cdot n \cdot q^{n-1}$, where d is the degree of P. Thus, when $q \geq 2dn$, the dWPHP axiom applies to show that there exists a point $\overline{a}_0$ in $[q]^n$ that is not in the range root of $f(\overline{a})$. This $\overline{a}_0$ is thus a non-root of P in the set $[q]^n$, like we wanted.

Next we show how to use this fact to get a hitting set with a second application of the dWPHP axiom. Let $\mathscr{C}$ be a class of algebraic circuits with n variables, syntactic degree

at most d, and size at most s. Consider the function

$$\begin{array}{rcccl} g & : & \mathscr{C} \times [q]^n \times \mathsf{codes}^r & \to & \mathsf{roots}^r \\ & & (P, \overline{a}, \overline{c}_1, \ldots, \overline{c}_r) & \mapsto & (f(\overline{a})(\overline{c}_1), \ldots, f(\overline{a})(\overline{c}_r)), \end{array} \tag{1}$$

where $\overline{c}_1, \ldots, \overline{c}_r$ are candidate codes, each from the code-set codes of size $n \cdot d \cdot q^{n-1}$, and $\overline{a}$ is a potential non-root of P in $[q]^n$. The parameter r should be sufficiently big, but polynomial. Then, it follows by construction and the fact proved in the previous paragraph that *any point outside the range of g is a hitting set*, since it will have a non-root for every algebraic circuit in $\mathscr{C}$. To find the point outside the range of g we invoke the dWPHP axiom, using again the assumption that $q \geq 2dn$.

One immediate consequence of Theorem 1.4 is that the theory $\mathsf{S}^1_2 + \mathsf{dWPHP}(\mathsf{PV})$ proves that the problem of verifying polynomial identities PIT can be solved by polynomial-size Boolean circuits, so is in $\mathsf{P/poly}$. We read this as adding evidence to the claim that the theory is sufficiently powerful to prove most contemporary results in complexity theory. In particular, it adds interest to the question of proving that the major lower bound conjectures of computational complexity are consistent with $\mathsf{S}^1_2 + \mathsf{dWPHP}(\mathsf{PV})$ and stronger theories; see [27, 10, 5] for more on this line of work.

1.3.3 Contribution to Reverse Mathematics of Complexity Theory

The fact that the dual weak pigeonhole principle suffices to prove the existence of small hitting sets raises a natural question: Is it also necessary? A positive answer would provide a *combinatorial* characterization of the *algebraic* statement that small hitting sets exist. It would also shed light on the role or the necessity of the probabilistic method in the usual proof of this existential statement. We show how to achieve a version of these two goals.

We define a formal scheme of *hitting sets axioms* called $\mathsf{HS}(\mathsf{PV})$. We follow two provisos. First, in view of the generality of Theorem 1.4, we define the axiom scheme to contain one axiom for each definable class $\mathscr{C}$ of algebraic circuits; the axiom states that each slice $\mathscr{C}_n$, consisting of the circuits of $\mathscr{C}$ with n variables, has small hitting sets. Second, in the definition of the axiom for $\mathscr{C}$, we need to decide whether to let it state the existence of hitting sets of unspecified but polynomial size, or to let it state the existence of hitting sets of some specified polynomial size. The bound established in Theorem 1.4 is actually of the form $\mathsf{poly}(m, n)$ where m is the logarithm of the number of circuits in the n-th slice, and $\mathsf{poly}(m, n)$ refers to a fixed polynomial of m and n. This dependence on m is common in most proofs of existence by the union bound. While the claim that hitting sets of any possibly larger but unspecified size exist would of course be also true, it turns out that asking the axiom to provide a hitting set of some fixed polynomial bound seems crucial in the proof of necessity of $\mathsf{dWPHP}(\mathsf{PV})$ that we are after. We chose the latter because this is what is sufficient, and it is still natural.

These two provisos motivate the following definition (informal; see Definition 5.1):

$\mathsf{HS}(\mathsf{PV})$: The set of *hitting set* axioms $\mathsf{HS}(g)$, for every $g \in \mathsf{PV}$. This states that if g defines a class $\mathscr{C}$ with its n-th slice having 2^m algebraic circuits with n variables, polynomial

degree, and polynomial size, then there is a hitting set for $\mathscr{C}$ over $[q]$ of size $\mathsf{poly}(m, n)$, with $q = \mathsf{poly}(n)$.

With the right definitions in place we can state the theorem that characterizes the proof-theoretic strength of the existence of small hitting sets:

Theorem 1.5 (Reverse Mathematics of Hitting Sets; informal, see Theorem 5.2)**.** *The axioms schemes* $\mathsf{dWPHP}(\mathsf{PV})$ *and* $\mathsf{HS}(\mathsf{PV})$ *are provably equivalent over the theory* S^1_2.

As remarked earlier, the sufficiency claim follows from Theorem 1.4. To prove the necessity we show how to use a hitting set to find a point outside the range of any given polynomial-time function $f : [N] \to [2N]$. To do this we design a class $\mathscr{C}_f$ of N many low-degree algebraic circuits each vanishing on the appropriate representation of a point in the image of f. We do so in such a way that a hitting set for $\mathscr{C}_f$ will correspond to an element in $[2N] \setminus \mathrm{Img}(f)$, completing the proof. To make this actually work we need to use a technique known as *amplification*, which goes back to the work of Paris-Wilkie-Woods [37]. The same kind of technique was discovered even earlier, in cryptography, to build pseudorandom number generators from hardcore bits; see the work of Blum-Micali [7]. The details of this argument can be found in Section 5 and Appendix A.

1.3.4 Application to Computational Complexity and Range Avoidance Problem

The proof-sketch we gave for Theorem 1.5 reveals a two-way connection between the computational problem of finding hitting sets and the so-called *Range Avoidance Problem*, or Avoid. The latter problem asks to find a point outside the range of a given function $f : [N] \to [2N]$. In recent years, Avoid has been studied with competing names. It was first studied by Kleinberg-Korten-Mitropolsky-Papadimitriou [24], calling it 1-Empty, and later by Korten [25], renaming it Empty. Those works defined it as the canonical complete problem for a complexity class of total search problems they called APEPP. Ren-Santhanam-Wang [41] studied it too, calling it Avoid, in their range avoidance problem approach to circuit lower bounds.

Our new coding-based proof of the existence of hitting sets shows that its associated search problem is in APEPP. The proof of Theorem 1.5 yields its completeness in the class:

Theorem 1.6 (Completeness of Finding Hitting Sets; informal, see Theorem 1.6)**.** *The total search problem that asks to find witnesses for the hitting set axioms of the scheme* $\mathsf{HS}(\mathsf{PV})$ *is* APEPP*-complete under* P^{NP}*-reductions.*

Perhaps not too surprisingly, the proof of Theorem 1.6 is almost identical to the proof of Theorem 1.5. Indeed, the necessity for the NP-oracle in the reductions comes (only) from the use of the amplification technique in the proof, as in earlier uses of this method; cf. [25].

A handful of complete problems for APEPP were known before, and some required P^{NP}-reductions too (cf. [24, 25]). But, to our knowledge, none of these complete problems related to the problem of constructing hitting sets for algebraic circuits. Here we used the new constructive proof of the Schwartz-Zippel Lemma to find an example of this type.

2 Preliminaries

2.1 Theories of Bounded Arithmetic

Here we define the formal theories we work with. Our main results are proved in the theories S^1_2 and its extension $\mathsf{S}^1_2 + \mathsf{dWPHP}(\mathsf{PV})$. These are single-sort theories for arithmetic, namely, number theory over $\mathbb{N}$. The theory S^1_2 is the first in Buss' hierarchy of theories of bounded arithmetic, and defines precisely the polynomial-time computable functions [8]. The theory $\mathsf{S}^1_2 + \mathsf{dWPHP}(\mathsf{PV})$ denotes its extension with the Dual Weak Pigeonhole Principle (dWPHP) for polynomial-time functions (PV), which allows to reason about probabilities and do approximate counting, based on the work of Jeřábek [21]. For a rigorous yet concise survey on bounded arithmetic the reader is referred to Buss [9] (for a full treatment of the area see [13, 26]; as well as a contemporary survey by Oliveira [33]). We refer the reader to Tzameret and Cook [46] for treatment of algebraic circuits over the integers inside theories of bounded arithmetic. Here we use a similar treatment of algebraic circuits, discussed in the next section.

The language $\mathcal{L}$. The language $\mathcal{L}$ contains the function symbols $+, \times, \lfloor \cdot/2 \rfloor, |\cdot|, \#, 0, 1$, and the relation symbol $\leq$. A finite collection of axioms called BASIC gives these symbols their intended meaning: 0 and 1 are constant symbols for the numbers 0 and 1, the symbols $+$ and $\times$ denote addition and multiplication of numbers, and the unary functions $\lfloor \cdot/2 \rfloor$ and $|\cdot|$ denote *rounded halving* and *binary length*, respectively. Precisely, if x is a natural number, then $\lfloor x/2 \rfloor$ is the largest natural number bounded by $x/2$, and $|x|$ is the length of x written in binary notation, except for $x = 0$ where $|x|$ is defined as 0 by convention. I.e., $|x| = \lceil \log_2(x+1) \rceil$. We say that $|x|$ is the *length of* x.

The only non-self-explanatory symbol is the "smash" binary function symbol # introduced by Buss [8]. The intention of $x\#y$ is 2 raised to the power of *the length* of x times *the length* of y, namely $2^{|x|\cdot|y|}$. This function allows us to work with strings as follows: assume we wish to talk about a binary string S of length n in the theory. Since the theory only talks about numbers, we represent the string as the number x between 2^{n-1} and $2^n - 1$ whose unique binary representation of length n is S. Note that the length n of S is precisely $|x|$. To consider a polynomial growth rate in the length of S, we need to be able to express strings of length n^c, for a constant c. Strings of length at most n^c are encoded, as before, by numbers of magnitude less than

$$2^{n^c} = 2^{|x|^c} = 2^{\overbrace{|x|\cdot|x|\cdots|x|}^{c \text{ times}}} = x\#(x\#\cdots\#(x\#x)) \ [c \text{ times}]. \tag{2}$$

We write $n \in \mathrm{Log}$ to mean that n serves as the length of some string, namely, $n = |x|$ for some x. Note that in bounded arithmetic theories where exponentiation may not be total, the formula $\forall n \exists x \ |x|{=}n$ is not provable. We use $\forall n{\in}\mathrm{Log}\ \psi$ and $\exists n{\in}\mathrm{Log}\ \psi$ as shorthand notations for the formulas $\forall x \forall n \ (|x|{=}n \to \psi)$ and $\exists x \exists n \ (|x|{=}n \wedge \psi)$.

The Theory $\mathsf{S}^1_2(\mathsf{PV})$. We need to define several polynomial-time computable functions in the theory and show that the theory proves some basic properties of these functions. We work in the extension of the language $\mathcal{L}$ with the language PV which has a new symbol for every polynomial-time computable function (to be precise, it has a new symbol for every function defined by Cobham's bounded recursion on notation [12]). The theory will be Buss's S^1_2 extended with the axioms defining the added PV-symbols ([8]; see [9]). This theory is sometimes denoted $\mathsf{S}^1_2(\mathsf{PV})$ but we will use the shorter notation S^1_2 if this does not lead to confusion as $\mathsf{S}^1_2(\mathsf{PV})$ is a conservative extension of S^1_2 for statements in the $\mathcal{L}$-language. For the precise definition of $\mathsf{S}^1_2(\mathsf{PV})$ see Krajíček [26, page 78].

Objects like strings, circuits, etc. in the theory are coded by numbers (cf. [8, 26]). In this sense, we may refer to an object in the theory, meaning formally its code. The inputs and outputs of PV-functions are natural numbers. However, the polynomial-time algorithms/Turing machines that compute the functions as PV-symbols get their inputs presented in binary notation as binary strings. The statement that n is an argument that is presented to a PV-function in unary notation means that it is expected that $n \in \mathrm{Log}$ (so 2^n exists) and that the argument is actually 2^n, so in this case the binary representation $100\cdots0$ of 2^n is given in the input of the polynomial-time algorithm that computes the function. An alternative convention would be to think of PV-functions as getting inputs of two *sorts*: numbers in unary notation and numbers in binary notation. This is the approach taken in the *two-sorted theories* of bounded arithmetic [13].

Definable sets and set-bounded quantification. Let Φ be a class of formulas in the language $\mathcal{L}$. A set of natural numbers $S \subseteq \mathbb{N}$ is called Φ-definable (with parameters) if there exists a formula $\varphi(x;y)$ in Φ, with all free variables indicated, such that for some $c \in \mathbb{N}$ we have $S = \{a \in \mathbb{N} : \mathbb{N} \models \varphi(a;c)\}$. This is definability in the *meta-theory*, or *in the standard model*. In a formal theory T such as S^1_2, a formula $\varphi(x;y)$ defines a set in every model M of the theory for every choice of the parameter $c \in M$ in the model:

$$[\varphi(x;c)]^M := \{a \in M : M \models \varphi(a;c)\}. \tag{3}$$

When the model or the parameter are not specified, a *definable set* is simply given by the formula $\varphi(x;y)$ that *defines* it (parametrically in y), and we write $[\varphi(x;y)]$.

For example, if we take $\varphi(x;y) := 1{\leq}x \wedge x{\leq}y$, then for every standard $c \in \mathbb{N}$ we have

$$[1{\leq}x \wedge x{\leq}c]^{\mathbb{N}} = [c] := \{1, 2, \ldots, c\}.$$

When this is not confusing, we use the same notation $[c] := \{1, 2, \ldots, c\}$ in the theory instead of the more accurate $[1{\leq}x \wedge x{\leq}c]$, even when c is a free variable. Also, we use $x{\in}[c]$ as short-hand notation for the formula $1{\leq}x \wedge x{\leq}c$. More generally, if we declare that a formula $\varphi(x;c)$ *defines a set which we denote* S_c, then we are entitled to use the following short-hand notations:

$$\begin{array}{lcl} x{\in}S_c & \equiv & \varphi(x;c) \\ \exists x{\in}S_c\ \psi & \equiv & \exists x\ (\varphi(x;c) \wedge \psi) \\ \forall x{\in}S_c\ \psi & \equiv & \forall x\ (\varphi(x;c) \rightarrow \psi). \end{array} \tag{4}$$

When S_c is a bounded set, i.e., there exists a term $t(c)$ such that every $x \in S_c$ satisfies $x \leq t(c)$ provably in the theory, then the quantifiers in (4) can be bounded: $\exists x{\leq}t(c)\ (\varphi(x;c) \wedge \psi)$ and $\forall x{\leq}t(c)\ (\varphi(x;c) \to \psi)$.

Dual Weak Pigeonhole Principle. Let f be a PV-symbol. The axiom dWPHP(f) is the universal closure of the following formula with free variables a, b, c:

$$\mathsf{dWPHP}^a_b(f_c) := (b{\geq}2a \wedge a{\geq}1 \to \exists y{\in}[b]\ \forall x{\in}[a]\ f_c(x) \neq y), \tag{5}$$

where $f_c(x)$ is notation for $f(x,c)$, thinking of c as a *parameter* for f. We write dWPHP(PV) for the axiom-scheme that contains all axioms dWPHP(f) for all PV-symbols f.

The presence of the parameter c in f_c makes the statement of dWPHP(f) more expressive. For example, the parameter may specify the sizes a and b of the intended domain and range of a function $f_c : [a] \to [b]$, say as $c = \langle a, b\rangle$, for an appropriate pairing function $\langle\cdot,\cdot\rangle$. Another example of the power of parameters is the following. The class of PV-functions admits a natural *universal-like* function, that we call eval, which provably in S^1_2 evaluates any PV function f: the theory S^1_2 proves $\forall b\ \forall x{<}b\ \ \mathrm{eval}(C_f(b), x){=}f(x)$ for a certain natural PV-function C_f. It follows from this that the infinite axiom scheme dWPHP(PV) is equivalent, over S^1_2, to the single axiom dWPHP(eval).

2.2 Polynomials and Algebraic Circuits

Let $\mathbb{G}$ be a ring. Denote by $\mathbb{G}[\overline{x}]$ the ring of (commutative) polynomials with coefficients from $\mathbb{G}$ and variables (indeterminates) $\overline{x} := \{x_1, x_2, \dots\}$. A polynomial is a formal linear combination of monomials, whereas a monomial is a product of variables. Two polynomials are identical if all their monomials have the same coefficients. The (total) degree of a monomial is the sum of all the powers of variables in it. The (total) degree of a polynomial is the maximum (total) degree of a monomial in it. The degree of an individual variable in a monomial is its power, and in a polynomial it is its maximum degree in the monomials of the polynomial. The maximum individual degree of a polynomial is the maximum degree of its variables.

Algebraic circuits and formulas. Algebraic circuits and formulas over the ring $\mathbb{G}$ compute polynomials in $\mathbb{G}[\overline{x}]$ via addition and multiplication gates, starting from the input variables and constants from the ring. In the rest of this paper the ring $\mathbb{G}$ will be fixed to the integers $\mathbb{Z}$.

An *algebraic circuit* (with parameters) is a directed acyclic graph (DAG) with its nodes labelled. The sources of the DAG are labelled by the name of an input. The internal nodes are labelled by gates of types $+$ and $\times$. The inputs of the circuit are split into variables and parameters; a parameter-free circuit is one without parameter inputs. The circuit may come with an implicit integer parameter assignment, in which case the corresponding sources of the DAG are labelled by the corresponding integer. We say that the constants are *plugged* into the circuit. An *algebraic formula* is an algebraic circuit whose DAG is a tree.

Size and syntactic degree. The representation size of a circuit is the number of bits that are needed to represent the DAG, the operation of each internal node of the graph, and the names of the variables and the parameters. If the circuit comes with implicit integer parameter assignment, then the representation size also includes the number of bits in their binary representations.

An algebraic circuit C can also be understood as a *straight-line program*, which is a finite sequence $C_0, C_1, \ldots, C_t$ of labelled *gates*. Each gate C_i is of one of four types: (1) a variable input gate, which is labelled by the name of a variable, or (2) a parameter input gate, which is labelled by the name of a parameter and perhaps also with an integer constant if the circuit comes with an implicit parameter assignment, or (3) an addition gate, which is labelled by the symbol $+$ and two integers $j < i$ and $k < i$ that specify the two operands in the addition $C_j + C_k$ that is computed at the gate, or (4) a multiplication gate, which is labelled by the symbol $\times$ and two integers $j < i$ and $k < i$ that specify the two operands in the multiplication $C_j \times C_k$ that is computed at the gate. The last gate C_t is the *output* of the circuit.

The *syntactic degree* d_i of the gate C_i is defined inductively on i: if C_i is a variable gate or a parameter gate, then $d_i = 1$; if C_i is an addition gate $C_j + C_k$, then $d_i = \max\{d_j, d_k\}$; and if C_i is a multiplication gate $C_j \times C_k$, then $d_i = d_j + d_k$. The *syntactic degree of* C is the syntactic degree of its output gate C_t. By induction on the number of gates in the circuit, it is easy to see that the syntactic degree of a circuit with t gates is at most 2^t. For formulas, a better upper bound is t.

A refinement of the concept of syntactic degree is *syntactic individual degree*. Suppose that C, with straight-line program $C_0, C_1, \ldots, C_t$, has n variables and m parameters. Let $u \in \{1, \ldots, n+m\}$ be the name of a variable or a parameter. The *syntactic individual degree of gate* C_i *on* u, denoted by $d_{i,u}$, is defined also inductively on i: If C_i is a variable or parameter gate with label u, then $d_{i,u} = 1$; if C_i is a variable or a parameter gate with label $v \neq u$, then $d_{i,u} = 0$; if C_i is an addition gate $C_j + C_k$, then $d_{i,u} = \max\{d_{j,u}, d_{k,u}\}$; if C_i is a multiplication gate $C_j \times C_k$, then $d_{i,u} = d_{j,u} + d_{k,u}$. The *syntactic individual degree of* C *on* u is $d_{t,u}$. Clearly, the syntactic degree of C is bounded by the sum of the individual degrees, and each individual degree is bounded by the syntactic degree. What we call syntactic degree is sometimes called syntactic *total* degree.

3 Schwartz-Zippel Lemma in the Theory

3.1 Notation and formalisations

Here we discuss formalisation in the theory and fix some notation. For every natural number $n \in \mathbb{N}$ we write $[n] = \{1, \ldots, n\}$. We fix some standard and efficient encoding for pairs, tuples, and lists of natural numbers as natural numbers, with its usual properties provable in the theory (cf. [8]). We discussed already in Section 2.1 how binary strings are encoded in the theory. For $n \in \mathrm{Log}$, we write $\{0,1\}^n$ for the definable set of strings of length n, with n as a parameter.

Ring of integers in the theory. Integers are encoded in the theory in the sign-magnitude representation as pairs (b, m) where $b \in \{0, 1\}$ is the sign, and m is the magnitude. The intention is that the pair $(b, m) \in \{0, 1\} \times \mathbb{N}$ encodes the integer $(-1)^b \cdot m$. Note that 0 has two codes. The *bit-complexity* of the integer is the length $|m|$ of its magnitude. When fed as argument into an algorithm operating with strings, the integer represented by (b, m) is presented as the pair (b, m) itself, with m written in binary notation. The set of codes (b, m) of integers in the theory is definable by the quantifier-free formula $(b{=}0 \vee b{=}1)$, and is denoted by $\mathbb{Z}$. The addition, subtraction, and multiplication operations on $\mathbb{Z}$ are PV-functions. The ordering relation $\leq$ on $\mathbb{Z}$ is a PV-predicate. The basic properties of these symbols are provable in the usual theories.

Algebraic circuits in the theory. Algebraic circuits and formulas over $\mathbb{Z}$ are formalised in the theory as labelled graphs with the integer constants that may appear on its leaves encoded in the sign-magnitude representation discussed above. We refer the reader to [46, Section 3.1.1] for more details on encoding algebraic circuits over $\mathbb{Z}$ in the theory. We define some PV-functions that operate with codes of algebraic circuits:

- size(C): the PV-function that computes the number of gates of the algebraic circuit C. In any reasonable explicit encoding of graphs we have size$(C) \leq |C|$.
- dimension(C): the PV-function that computes the number of variables, or indeterminates, of the algebraic circuit C, called the *dimension* of C.
- parametric-dimension(C): the PV-function that computes the number of parameters of the algebraic circuit C, called the *parametric dimension* of C.
- total-degree(C): the PV-function that computes the *syntactic* total degree of the algebraic circuit C. Recall that this is bounded by $2^{\text{size}(C)}$, hence by $2^{|C|}$, so its binary representation fits in $|C|$ bits. Recall also that our definition of syntactic degree counts the parameter inputs as contributing to the degree. One consequence of this is that if the circuit is fed with integers for its n variables, and integers for its m parameters, all of bit-complexity at most s, then the output has bit-complexity polynomial in d, s, n, m, where d is the total degree.
- individual-degree(C, i): the PV-function that computes the *syntactic* individual degree of the i-th input in the algebraic circuit C, where $1 \leq i \leq n + m$, and n and m are the dimension and the parametric dimension of C. The *maximum syntactic individual degree* of the circuit is denoted max-individual-degree(C). When C has a single variable, we write degree(C) instead of max-individual-degree(C).
- eval-arithmetic$(C, \overline{a}, \overline{p}, d)$: the PV-function for the standard polynomial-time algorithm which, given an integer d in unary notation, given an algebraic circuit C of syntactic maximum individual degree d, given vectors of integer inputs $\overline{a} = (a_1, \ldots, a_n)$ for the variables, and $\overline{p} = (p_1, \ldots, p_m)$ for the parameters, evaluates C on inputs $\overline{a}$ and $\overline{p}$, with the gates that are labelled by $+$ and $\times$ interpreted as the addition and multiplication operations of the integers. This is defined inductively on the structure of C

and the standard algorithm runs in time polynomial in the size of its input because the syntactic degree d is given in unary notation. Here we use also the fact that the definition of syntactic degree takes the parameter inputs into account. In other words, this PV-function models the evaluation of algebraic circuits of polynomial-degree with constants of polynomial bit-complexity. See [46, Sec. 11.1] how to define this function in PV (first balancing the algebraic circuit of polynomial syntactic-degree, and then evaluating the balanced circuit; in PV however, the initial balancing step is not necessary). If the parameter inputs $\overline{p}$ are plugged into C, then the notation $C(\overline{a})$ abbreviates eval-arithmetic$(C, \overline{a}, \overline{p}, \text{total-degree}(C))$.

We need also some notation for *definable sets of circuits*; see Section 2.1 for a discussion on definability in the theory. With the encodings discussed so far, all the sets below are quantifier-free definable in the language $\mathcal{L}$.

- Ckt: the set of (codes of) algebraic circuits. The subset of formulas is denoted Fms.
- Ckt(n, d): the set of (codes of) algebraic circuits with at most n indeterminates and syntactic maximum individual degree at most d. The subset of formulas is denoted Fms(n, d).
- Ckt(n, d, s): the subset of Ckt(n, d) whose elements have representation size at most s. The subset of formulas is denoted Fms(n, d, s).
- UniPoly(d): the set of (codes of) univariate polynomials with integer coefficients and degree at most d, written (as formulas) in explicit sum of monomials form. In other words, if x denotes the indeterminate, then UniPoly(d) denotes the set of polynomials of the form $P(\mathrm{x}) = c_0 + c_1\mathrm{x} + c_2\mathrm{x}^2 + \cdots + c_d\mathrm{x}^d$, where, for $i = 0, \ldots, d$, the i-th coefficient c_i is an integer. Note that when we say *degree at most* d we do *not* require that $c_d \neq 0$; that would be degree *exactly* d.
- UniPoly(d, s): the subset of UniPoly(d) in which all coefficients are integers of bit complexity at most s; i.e., each coefficient c_i is an integer in the interval $[-2^s+1, 2^s-1]$. We write coef(P, i) for the PV-function that extracts the coefficient of the term of degree $i \in \{0, \ldots, d\}$ of the univariate polynomial $P \in$ UniPoly(d). This is trivially computable in polynomial time because the polynomials in UniPoly(d) are given in explicit sum of monomials form.

Finally, we need to define the concept of *semantic equivalence over a definable set*. If F and G denote algebraic circuits with the same number n of indeterminates, and S is a (finite or infinite) definable set of integers, with its membership predicate $x \in S$ definable by a formula (with or without parameters), then the notation $F \equiv_S G$ stands for the (possibly unbounded) formula

$$\forall \overline{a} \in S^n \; F(\overline{a}) = G(\overline{a}). \tag{6}$$

In words, this formula says that evaluations of F and G agree on every n-vector of integers $\overline{a} = (a_1, \ldots, a_n)$ in S^n. If q is an integer (in the theory), then we use S_q to denote the definable set $\{0, 1, \ldots, q-1\}$ of integers between 0 and $q-1$; i.e.,

$$S_q := \{0, 1, \ldots, q-1\} \subseteq \mathbb{Z} \tag{7}$$

We write $F \equiv_q G$ instead of $F \equiv_{S_q} G$, and the corresponding formula can be written with bounded quantifiers: $\forall \overline{a} \leq q\ (\overline{a} \in S_q^n \to F(\overline{a}) = G(\overline{a}))$ or, more precisely, the bounded quantifier is $\forall \overline{a} \leq t(n,q)$, where $t(n,q)$ is the PV-function that bounds the encodings of the elements of S_q^n. In the other extreme case in which S is the set of *all* integers $\mathbb{Z}$, we write $F \equiv S$, and the corresponding formula has unbounded quantifiers $\forall \overline{a} \in \mathbb{Z}^n\ F(\overline{a}) = G(\overline{a})$.

Definable classes of algebraic circuits. We introduce next the concept of a *definable class of algebraic circuits*. The intention is to capture the usual practice in algebraic circuit complexity of considering different subclasses of circuits with varying parameters (cf. [43]). Examples include: formulas, syntactic multilinear circuits, uniform families, projections of the determinant polynomial, arithmetizations of Boolean circuits, algebraic branching programs, etc. We have already seen two classes of definable classes: the class of circuits Ckt, and the class of formulas Fms. The following definition generalizes these:

Definition 3.1 (In S^1_2)**.** A *definable class of algebraic circuits* is a subset $\mathscr{C} \subseteq \mathrm{Ckt}$ of (codes of) algebraic circuits that comes with a PV-function g, the *decoding function*, that is surjective onto $\mathscr{C}$. The polynomial-time algorithm that computes g as a PV-function must satisfy the following condition: given n, d, s in unary and given a string $x \in \{0,1\}^m$, the function $g_e(n, d, s, x)$ outputs an algebraic circuit in $\mathrm{Ckt}(n, d, s)$, for all settings of the parameter e.

For the rest of this section, let $\mathscr{C}$ be a definable class of algebraic circuits with decoding function g. If C is an algebraic circuit in $\mathscr{C}$ and $g_e(n, d, s, x) = C$, then we say that x *is a description of* C *as a member of* $\mathscr{C}$. The *description size of* C *as a member of* $\mathscr{C}$ is the length as a string of its smallest description as a member of $\mathscr{C}$.

We write $\mathscr{C}_e(n, d, s, m)$ for the *slice* of algebraic circuits in $\mathscr{C}$ of the form $g_e(n, d, s, x)$ with $x \in \{0,1\}^m$; in symbols:

$$\mathscr{C}_e(n, d, s, m) := \{g_e(n, d, s, x) : x \in \{0,1\}^m\}.$$

Note that $\mathscr{C}_e(n, d, s, m)$ is always a subset of $\mathrm{Ckt}(n, d, s)$, but its cardinality is at most 2^m, which may be much smaller than the cardinality of $\mathrm{Ckt}(n, d, s)$. When $2^m \ll 2^s$, we say that the class is *sparse*. An extreme case of this occurs when $\mathscr{C}$ is a class of algebraic circuits determined by their dimension, say $(C_n)_{n \geq 1}$ with $C_n \in \mathrm{Ckt}(n, d(n), s(n))$, where each $\mathscr{C}(n, d, s)$ has at most one member C_n. A typical example is the class of *determinant polynomials* $(\det_n)_{n \geq 1}$, where $\det_n$ denotes the polynomial with n^2 indeterminates that computes the determinant of the $n \times n$ matrix given by its n^2 inputs. A related but less extreme example of a sparse definable class is the class of determinants of Tutte matrices of graphs. In this example, the representation size of a member in this class is determined by the number of edges of the underlying graph.

3.2 Fundamental Theorem of Algebra: the Univariate Case

The Fundamental Theorem of Algebra (FTA) states a fundamental property of univariate polynomials over the field of complex numbers.

Theorem 3.2 (Fundamental Theorem of Algebra)**.** *Every non-zero univariate polynomial of degree d with coefficients in the field of complex numbers has exactly d roots in the complex numbers.*

The theorem naturally splits into two halves: the half that states that there are *at least d roots*, and the half that states that there are *at most d roots*. While the *at least* statement relies on special properties of the complex numbers, the proof of the *at most* statement relies only on the fact that the class of univariate polynomials with coefficients in a field forms a Euclidean domain where the Euclidean division algorithm applies. In particular, this means that the *at most* statement holds also for polynomials over any subring of a field; for example, polynomials over the ring of integers.

In this section we show that the theory S^1_2 is able to formalise the standard proof of the second half of the FTA for the class of univariate polynomials over the ring of integers. This will be used as a building block in the next section. While the underlying idea of this proof is completely standard, it is somewhat delicate to carry out the argument in S^1_2 because we need to keep control of the bit-complexity of the coefficients that appear along the way in the computations.

Let d and q be small non-negative integers. Let P be an element of UniPoly(d), that is, P is (the code of) a univariate polynomial with integer coefficients $c_0, c_1, \ldots, c_d$ of arbitrary bit-complexity, and degree at most d. If we write x for the indeterminate, then

$$P(\mathrm{x}) = c_0 + c_1\mathrm{x} + \cdots + c_d\mathrm{x}^d. \tag{8}$$

Define

$$\begin{array}{lcl} S_q & := & \{0, \ldots, q-1\} \subseteq \mathbb{Z}, \\ Z_{P,q} & := & \{u \in S_q : P(u) = 0\}. \end{array}$$

Note that $Z_{P,q}$ is the set of roots of $P(\mathrm{x})$ in the set S_q. The goal is to show that if P is not the zero polynomial, then $Z_{P,q}$ has cardinality at most d. For later reference, we state this upper bound in the form of the existence of a surjection from the set $[d]$ onto the set $Z_{P,q}$.

Remark 3.3. With P given as in (8), where the number of roots is polynomial in the representation size of P, we could also choose to state the upper bound by listing the roots. Indeed, provided d is polynomial in the representation size, this holds true even if P is given by a univariate algebraic circuit, as we will later see. However, in the multivariate case with $n > 1$ variables, the number of roots in S^n_q may be exponential in the representation size of P, even if d is small, so there we need to resort to the surjective mapping formulation of the upper bound. For this reason and to be able to reuse it, we state the upper bound for the univariate case with a surjective mapping too.

We define a PV-function

$$h_{P,q} : [d] \to S_q \cup \{q\}$$

by the polynomial-time algorithm that computes it. The input to the algorithm for $h_{P,q}(i)$ is the triple (P, q, i) with P as above, q given in unary notation, and $i \in [d]$. In the algorithm, let u loop over the set S_q and evaluate $P(\mathrm{x})$ at $\mathrm{x} = u$, keeping a counter of the number of distinct roots found along the way. If $P(\mathrm{x})$ has at least i distinct roots in S_q, then $h_{P,q}(i)$ is defined to be the i-th smallest root of $P(\mathrm{x})$ in S_q. Otherwise, $h_{P,q}(i)$ is set to the value q seen as an end-of-list marker. In other words, the sequence $(h_{P,q}(1), \ldots, h_{P,q}(d))$ equals $(r_1, \ldots, r_t, q, \ldots, q)$ where $r_1 < \cdots < r_t$ is the ordered list of the first d roots of $P(\mathrm{x})$ in S_q. The next lemma states that S^1_2 proves that if $P(\mathrm{x})$ does not vanish everywhere on S_q, then this enumeration indeed covers all the roots of $P(\mathrm{x})$ in S_q.

Lemma 3.4 (Second Half of Fundamental Theorem of Algebra in S^1_2)**.** *For all $d, q \in \mathrm{Log}$ and every degree-d polynomial $P \in \mathrm{UniPoly}(d)$, if not all coefficients of P are zero, then the sequence $h_{P,q}(1), h_{P,q}(2), \ldots, h_{P,q}(d)$ contains all the roots of P in S_q; i.e., the function $h_{P,q}$ is surjective onto $Z_{P,q}$.*

Proof. Fix $d, q \in \mathrm{Log}$ and let P be an element of $\mathrm{UniPoly}(d)$. Let s be the bit-complexity of the coefficients of P, so s is a length and $P \in \mathrm{UniPoly}(d, s)$. The plan for the proof is to define a sequence $s_0, s_1, \ldots, s_d$ of lengths with $s_d = s$, and then prove $\phi(z)$ by Π^b_1-PIND, where $\phi(z)$ is the following Π^b_1-formula:

$$\begin{aligned}\phi(z) \;:=\; & \forall k{\leq}z\ (k{=}|z|{\leq}d \rightarrow \forall A{\in}\mathrm{UniPoly}(k, s_k) \\ & \qquad (\exists i{\leq}k\ \mathrm{coef}(A, i){\neq}0) \rightarrow (\forall u{\in}Z_{A,q}\ \exists i{\leq}k\ h_{A,q}(i){=}u)).\end{aligned}$$

Observe that the conclusion of the lemma is the specialization of $\phi(2^d{-}1)$ to $k := d$ and $A := P$. The formula $\phi(z)$ uses d, q and the (code of the) sequence $s_0, \ldots, s_d$ as parameters. Note also that $\phi(z)$ is indeed a Π^b_1-formula because $d, q \in \mathrm{Log}$ and, therefore, all quantifiers except the first two are sharply bounded: for $\exists i{\leq}k$, note the condition $k{=}|z|$; for $\forall v{\in}Z_{A,q}$ recall that $Z_{A,q} \subseteq S_q$, and S_q is the set of codes of integers $\{0, 1, \ldots, q-1\} \subseteq \mathbb{Z}$, and these are bounded by $2^{|q|+1} \leq 4q$ (using sign-magnitude representation).

We still need to define s_k for $k = 0, \ldots, d$. We define s_k by the following inverse recurrence relation for $k = d, d-1, \ldots, 2, 1$:

$$\begin{aligned} s_d \;&:=\; s, \\ s_{k-1} \;&:=\; s_k + k|q| + |k|. \end{aligned} \tag{9}$$

Since Log is provably closed under addition and multiplication and d is a length, by Π^b_1-PIND each s_k is a length. Also by Π^b_1-PIND, each s_k is bounded by $s{+}d^2|q|{+}d^2$. Since in the rest of the proof the parameter q will be fixed, we write Z_A and S instead of the more accurate $Z_{A,q}$ and S_q. Similarly, we write h_A instead of $h_{A,q}$.

To prove $\phi(2^d{-}1)$ we proceed by Π^b_1-PIND. Concretely, we prove $\forall k{<}d(\phi'(k) \rightarrow \phi'(k{+}1))$ and $\phi'(0)$, where $\phi'(k)$, with k a length, is shorthand notation for $\phi(2^k - 1)$,

Base case: $\phi'(0)$. In this case we have $A(\mathrm{x}) = a_0$ for a single integer a_0 and, therefore, either $a_0 = 0$, or else $Z_A = \emptyset$ because $a_0 \neq 0$ and hence h_A is vacuously surjective onto Z_A.

Inductive step: $\forall k{<}d\ (\phi'(k) \to \phi'(k{+}1))$. For convenience of notation, we shift the induction parameter k by one unit, which is clearly equivalent: assuming $1 \le k \le d$ and $\phi'(k-1)$, we prove $\phi'(k)$. Fix A in UniPoly(k, s_k), say $A(\mathrm{x}) = a_0 + a_1\mathrm{x} + \cdots + a_k\mathrm{x}^k$. Assume also that not all coefficients $a_0, a_1, \ldots, a_k$ are zero. When $Z_A = \emptyset$ there is nothing to prove as then h_A is vacuously surjective onto Z_A. Assume then that $Z_A \neq \emptyset$ and let then $v \in S$ be such that $A(v) = 0$. We define the coefficients $b_0, b_1, \ldots, b_{k-1}$ of a degree-$(k-1)$ polynomial $B(\mathrm{x}) = b_0 + b_1\mathrm{x} + \cdots + b_{k-1}\mathrm{x}^{k-1}$ and show that

$$A(\mathrm{x}) \equiv (\mathrm{x} - v)B(\mathrm{x}). \tag{10}$$

This will imply that $Z_A = Z_B \cup \{v\}$, so the list of roots of A will be easily obtained from the list of roots of B.

The coefficients $b_0, \ldots, b_{k-1}$ of $B(\mathrm{x})$ are defined by the inverse recurrence relation:

$$\begin{array}{lcl} b_k & := & 0, \\ b_{k-i} & := & a_{k-i+1} + b_{k-i+1}v, \end{array} \tag{11}$$

for $i = 1, \ldots, k$.

Claim 3.5. *The polynomial $B(\mathrm{x})$ belongs to* UniPoly$(k-1, s_{k-1})$.

Proof of Claim 3.5. Clearly, there exists a PV-function that, given the appropriate inputs, computes the sum of the sequence $a_{k-i+j}v^{j-1}$ for $j = 1, \ldots, i$. Using this, by quantifier free induction on $i = 0, \ldots, k$ we have

$$b_{k-i} = a_{k-i+1} + a_{k-i+2}v + a_{k-i+3}v^2 + \cdots + a_k v^{i-1}. \tag{12}$$

For all $j = 1, \ldots, i$, the j-th term in (12) has bit-complexity bounded by $s_k + k|q|$: indeed a_{k-i+j} has bit-complexity at most s_k, and $v^{j-1} < q^k$. Thus, each b_{k-i} has bit complexity at most $s_k + k|q| + |k|$, which equals s_{k-1} by (9). Thus, the polynomial $B(\mathrm{x})$ given by the coefficients $b_0, b_1, \ldots, b_{k-1}$ is indeed an element of UniPoly$(k-1, s_{k-1})$. □

Claim 3.6. *Not all coefficients of the polynomial $B(\mathrm{x})$ are 0.*

Proof of Claim 3.6. By assumption, not all the coefficients $a_0, a_1, \ldots, a_k$ of $A(\mathrm{x})$ are zero. Therefore, by quantifier-free maximization, there is a largest $j \le k$ such that $a_j \neq 0$, so all coefficients $a_{j+1}, a_{j+2}, \ldots, a_k$ are 0. By (11) and quantifier-free reverse induction, this means that all coefficients $b_k, b_{k-1}, \ldots, b_j$ are also 0. Now we argue that $j \ge 1$. Indeed, if $j = 0$, then all bs are 0 so by (12) with $i = k$ we get $A(v) = a_0 + b_0 v = a_0$. But v was such that $A(v) = 0$, and a_0 is non-zero since $j = 0$; a contradiction. Thus $j \ge 1$. But then $b_{j-1} = a_j + b_j v = a_j \neq 0$, since $b_j = 0$. □

Claim 3.7. $A(\mathrm{x}) \equiv (\mathrm{x} - v)B(\mathrm{x})$.

Proof of Claim 3.7. Fix $u \in \mathbb{Z}$. Our goal is to show that $A(u) = (u - v)B(u)$. To see this first set the following notation:

$$\begin{array}{rcl} A_0(\mathrm{x}) & := & a_{k-1}\mathrm{x}^{k-1} + a_k\mathrm{x}^k, \\ A_i(\mathrm{x}) & := & a_{k-1-i}\mathrm{x}^{k-1-i} + A_{i-1}(\mathrm{x}), \end{array} \tag{13}$$

for $i = 1, \ldots, k-1$, and also

$$\begin{array}{rcl} B_0(\mathrm{x}) & := & b_{k-1}\mathrm{x}^{k-1}, \\ B_i(\mathrm{x}) & := & b_{k-1-i}\mathrm{x}^{k-1-i} + B_{i-1}(\mathrm{x}), \end{array} \tag{14}$$

for $i = 1, \ldots, k-1$. Note that $A_{k-1}(\mathrm{x})$ is just an alternative notation for $A(\mathrm{x})$, and $B_{k-1}(\mathrm{x})$ is an alternative notation for $B(\mathrm{x})$. With these, we show, by quantifier-free induction on $i = 0, \ldots, k-1$ with u, v, k and (the codes of) $A(\mathrm{x})$ and $B(\mathrm{x})$ as parameters, that

$$A_i(u) = (u - v)B_i(u) + (a_{k-1-i} + b_{k-1-i}v)u^{k-1-i}. \tag{15}$$

The base case $i = 0$ of this equation checks out as

$$\begin{aligned} A_0(u) &= a_{k-1}u^{k-1} + a_k u^k && (16)\\ &= (u - v)b_{k-1}u^{k-1} + (a_{k-1} + b_{k-1}v)u^{k-1}, && (17)\\ &= (u - v)B_0(u) + (a_{k-1} + b_{k-1}v)u^{k-1} && (18) \end{aligned}$$

where the first equality follows from the choice of $A_0(\mathrm{x})$ in (13), the second equality follows from the choice of b_{k-1} in (11), and the third equality follows from the choice of $B_0(\mathrm{x})$ in (14). Similarly, assuming (15) for $0 \leq i \leq k-2$ as induction hypothesis, the same equation for $i+1$ checks out as

$$\begin{aligned} A_{i+1}(u) &= a_{k-2-i}u^{k-2-i} + A_i(u) && (19)\\ &= a_{k-2-i}u^{k-2-i} + (u - v)B_i(u) + (a_{k-1-i} + b_{k-1-i}v)u^{k-1-i} && (20)\\ &= (u - v)(b_{k-2-i}u^{k-2-i} + B_i(u)) + (a_{k-2-i} + b_{k-2-i}v)u^{k-2-i}, && (21)\\ &= (u - v)B_{i+1}(u) + (a_{k-2-i} + b_{k-2-i}v)u^{k-2-i}, && (22) \end{aligned}$$

where the first equality follows from the choice of $A_{i+1}(\mathrm{x})$ in (13), the second equality follows from the induction hypothesis, the third equality follows from the choice of b_{k-2-i} in (11), and the fourth equality follows from the choice of $B_{i+1}(\mathrm{x})$ in (14). This gives

$$\begin{aligned} A(u) &= A_{k-1}(u) && (23)\\ &= (u - v)B_{k-1}(u) + (a_0 + b_0 v) && (24)\\ &= (u - v)B(u) + A(v) && (25)\\ &= (u - v)B(u) && (26) \end{aligned}$$

where the first equality follows from the choice of $A_{k-1}(\mathrm{x})$, the second follows from (15) with $i = k-1$, the third follows from the choices of $B_{k-1}(\mathrm{x})$ and b_0, and the fourth follows from the fact that v is a root of $A(\mathrm{x})$. Since u was an arbitrary value in $\mathbb{Z}$, this proves the claim. □

We are ready to complete the proof of the lemma. We argued via Claim 3.5 that B is an element of $\mathrm{UniPoly}(k-1, s_{k-1})$, and via Claim 3.6 that not all coefficients of B are zero. Therefore, the induction hypothesis $\phi'(k-1)$ applied to B gives that the function $h_B : [k-1] \to S \cup \{q\}$ is surjective onto Z_B. Think of the function h_B as a list $(r_1, \ldots, r_t, q, \ldots, q)$ of $t \leq k-1$ roots $r_1 < \cdots < r_t$ of B in S, followed by a string of end-of-list markers q. We argued via Claim 3.7 that $Z_A = Z_B \cup \{v\}$; recall for this that v was from S. Therefore, the list h_A of roots in S for A is obtained from h_B by inserting v in the right place of the list of at most $k-1$ roots of B, or ignoring it if it is already there, in such a way that the resulting list is the sorted list of distinct roots of A in S followed by end-of-list markers. It follows that h_A is surjective onto Z_A, as was to be shown. □

3.3 Coefficients of Univariate Polynomials

Next we show that the coefficients of the univariate polynomial computed by an algebraic circuit with a single indeterminate and with integer coefficients can be computed in time polynomial in the representation-size of the circuit, provably in S^1_2.

Consider the PV-function polynomial(C, d) given by the following polynomial-time algorithm: The input is a positive integer d written in unary notation, and an algebraic circuit C with a single indeterminate x and syntactic degree bounded by d. The circuit can be thought of as given by a straight-line program of gates $C_0, C_1, \ldots, C_t$, identified with their labels. For all gate numbers $u = 0, 1, \ldots, t$ in this order, the algorithm constructs the coefficients $a_{u,0}, \ldots, a_{u,d}$ of a degree-d polynomial $P_u \in \mathrm{UniPoly}(d)$, inductively according to the following rules:

1. if $C_u = c \in \mathbb{Z}$, then $a_{u,0} := c$ and $a_{u,k} := 0$ for $k = 1, 2, \ldots, d$,
2. if $C_u = \mathrm{x}$, then $a_{u,0} := 0$ and $a_{u,1} := 1$, and $a_{u,k} := 0$ for $k = 2, 3, \ldots, d$,
3. if $C_u = C_v + C_w$ with $v < u$ and $w < v$, then $a_{u,k} := a_{v,k} + a_{w,k}$ for $k = 0, 1, \ldots, d$,
4. if $C_u = C_v \times C_w$ with $v < u$ and $w < u$, then $a_{u,k} := \sum_{t=0}^{k} a_{v,t} a_{w,k-t}$ for $k = 0, 1, \ldots, d$.

On termination, the algorithm outputs the code of the polynomial P_t.

We verify that the algorithm runs in polynomial time. Let s be the bit-complexity of the constants plugged into the circuit C. By the definition of the representation-size of algebraic circuits, s is bounded by the representation-size of C. The bit-complexity s_u of the coefficients $a_{u,0}, \ldots, a_{u,d}$, and the syntactic degree d_u of the circuit rooted at u, obey the following recurrences:

1. $s_u \leq s$ and $d_u = 1$ if $C_u = c \in \mathbb{Z}$,
2. $s_u \leq 2$ and $d_u = 1$ if $C_u = \mathrm{x}$,
3. $s_u \leq 1 + \max\{s_v, s_w\}$ and $d_u \leq \max\{d_v, d_w\}$ if $C_u = C_v + C_w$,
4. $s_u \leq |d+1| + (s_v + s_w)$ and $d_u \leq d_v + d_w$ if $C_u = C_v \times C_w$.

Therefore, by induction on $u = 0, 1, \ldots, t$, we get $s_u \leq (s + |d+1|)(2d_u - 1)$. Since $d_u \leq d$, the conclusion is that each s_u is bounded by a polynomial $\mathsf{poly}(d, s)$, so the algorithm runs in time polynomial in the size of its input.

Lemma 3.8 (Univariate Coefficient-Extraction Lemma in S^1_2)**.** *For every $d \in \mathrm{Log}$, every algebraic circuit C with a single indeterminate input, syntactic degree at most d, and integer constants, we have $P \equiv C$, for $P = \mathrm{polynomial}(C, d)$.*

Proof. Let $C_0, C_1, \ldots, C_t$ be the straight-line program of C and let $\overline{p}$ be the integer constants plugged into C. Let $P_0, P_1, \ldots, P_t$ be the degree-d polynomials constructed by the algorithm that defines $\mathrm{polynomial}(C, d)$. For every $z \in \mathbb{Z}$, fixed as parameter of induction together with $\overline{p}, d$ and the sequences $C_0, C_1, \ldots, C_t$ and $P_0, P_1, \ldots, P_t$, prove $P_u(z) = C_u(z)$, i.e.,

$$\text{eval-arithmetic}(P_u, z, \overline{p}, d) = \text{eval-arithmetic}(C_u, z, \overline{p}, d), \tag{27}$$

by quantifier-free induction on $u = 0, 1, \ldots, t$. For this, use the rules 1–4 that define the algorithm of the PV-function $\mathrm{polynomial}(C, d)$. Note that this works for any value $z \in \mathbb{Z}$, regardless of its bit-complexity. Thus, applying (27) to $t = u$ we get $P(z) = P_t(z) = C_t(z) = C(z)$. Since z was arbitrary in $\mathbb{Z}$, this shows $P \equiv C$ and the lemma is proved. □

3.4 The Coding Argument and its Formalisation

We show that the theory S^1_2 proves the Schwartz-Zippel Lemma for multivariate polynomials of polynomial degree. As in the univariate case, in the n-dimensional case we consider the roots with all its entries in S_q for given q, where $S_q = \{0, 1, \ldots, q-1\}$. For any non-zero polynomial $P(\overline{\mathrm{x}})$ with n variables and individual degree at most d, any given non-root $\overline{a}$ of $P(\overline{\mathrm{x}})$, and any given positive integer q, we build a function that has domain-size $n \cdot d \cdot q^{n-1}$ and covers, in its range, all the roots of $P(\overline{\mathrm{x}})$ within S^n_q. Note that in this case the covering function must be given access to one non-root $\overline{a}$ of $P(\overline{\mathrm{x}})$ that certifies it as non-zero. In other words, the statement that the theory S^1_2 will formalise is the following:

> *if $P(\overline{\mathrm{x}})$ has n variables, individual degree d, and at least one non-root $\overline{a}$ in $\mathbb{Z}^n$, then $P(\overline{\mathrm{x}})$ has at least $q^n \cdot (1 - nd/q)$ non-roots in S^n_q.*

To formulate and prove this statement with the required precision, we need to introduce some notation.

Let n, d, q be small positive integers. Let P be an element of $\mathrm{Ckt}(n, d)$, that is, P is (the code of) an algebraic circuit with n inputs and syntactic individual degree at most d. As in the univariate case, define

$$\begin{array}{lcl} S_q & := & \{0, \ldots, q-1\} \subseteq \mathbb{Z}, \\ Z_{P,q} & := & \{(u_1, \ldots, u_n) \in S^n_q : P(u_1, \ldots, u_n) = 0\}, \\ C_{n,d,q} & := & [n] \times [d] \times S^{n-1}_q. \end{array}$$

Note that $Z_{P,q}$ is the set of roots, or *zeros*, of P in the set S_q^n. We will use $C_{n,d,q}$ as the set of *codes* of roots of P in the set S_q^n. Observe that if $q > nd$, then $|C_{P,d,q}| < |Z_{P,q}|$ and compression is achieved.

For the fixed n, d, q, P and any fixed additional vector $\overline{a} = (a_1, \ldots, a_n)$ of integers (not necessarily in S_q^n) we define a pair of PV-functions

$$\begin{aligned} \mathrm{enc}_{P,\overline{a}} &: S_q^n \to C_{n,d,q}, \\ \mathrm{dec}_{P,\overline{a}} &: C_{n,d,q} \to S_q^n. \end{aligned}$$

These functions are defined by the polynomial-time algorithms that compute them.

Encoding. The input to $\mathrm{enc}_{P,\overline{a}}(\overline{b})$ is a 5-tuple $(P, d, q, \overline{a}; \overline{b})$ with $P, d, q, \overline{a}$ as above, with d and q presented in unary notation, and $\overline{b} = (b_1, \ldots, b_n)$ a vector S_q^n. The goal is to encode the vector $\overline{b}$ as an element in $C_{n,d,q}$. The algorithm first evaluates P at $\overline{a}$ and $\overline{b}$ to check if $P(\overline{a}) \neq 0$ and $P(\overline{b}) = 0$. If this fails, then either the given $\overline{a}$ is not suitable for the encoding scheme, or the given $\overline{b}$ is not even a root, which we do not need to encode. In such a case the algorithm returns the default value $(1, 1, \overline{0})$ from $[n] \times [d] \times S_q^{n-1}$, where $\overline{0} = (0, \ldots, 0)$ is the all-zero vector in S_q^{n-1}. If indeed $P(\overline{a}) \neq 0$ and $P(\overline{b}) = 0$, then the algorithm loops through $k = 1, 2, \ldots, n$ in this order until it finds the first position where

$$\begin{array}{lcl} P(a_1, \ldots, a_n) & \neq & 0 \\ P(b_1, a_2, \ldots, a_n) & \neq & 0 \\ P(b_1, b_2, a_3, \ldots, a_n) & \neq & 0 \\ \vdots \quad \ddots & \vdots & \vdots \\ P(b_1, b_2, b_3, \ldots, b_{k-1}, \ a_k \ , a_{k+1}, \ldots, a_n) & \neq & 0 \\ P(b_1, b_2, b_3, \ldots, b_{k-1}, \ b_k \ , a_{k+1}, \ldots, a_n) & = & 0. \end{array} \tag{28}$$

A first such $k \in [n]$ must exist since $P(\overline{a}) \neq 0$ and $P(\overline{b}) = 0$. Next, consider the *univariate* algebraic circuit

$$Q(\mathrm{x}) := P(b_1, \ldots, b_{k-1}, \ \mathrm{x}, \ a_{k+1}, \ldots, a_n) \tag{29}$$

By definition $Q(b_k) = 0$ so b_k is a root of $Q(\mathrm{x})$. Then the algorithm loops through $r \in S_q$ to form the ordered list of roots of $Q(\mathrm{x})$ in S_q and finds the position $i \in [q]$ of b_k in this ordered list. If this index exceeds d (spoiler: this will never happen but the algorithm need not know this), then something went wrong and the algorithm outputs the default value $(1, 1, \overline{0})$ as before. Otherwise, it outputs $(k, i, b_1, \ldots, b_{k-1}, b_{k+1}, \ldots, b_n)$, which is in $[n] \times [d] \times S_q^{n-1}$.

Decoding. The input to $\mathrm{dec}_{P,\overline{a}}(k, i, \overline{c})$ is the 7-tuple $(P, d, q, \overline{a}; k, i, \overline{c})$, with $P, d, q, \overline{a}$ as above, with d and q presented in unary notation, $k \in [n]$ and $i \in [d]$, and $\overline{c} = (c_1, \ldots, c_{n-1})$ a vector in S_q^{n-1}. The algorithm first evaluates P on the vector $\overline{a}$ to check that $P(\overline{a}) \neq 0$. If this check fails, then the given $\overline{a}$ is not suitable for the encoding scheme. In such a case the algorithm outputs the default value $\overline{0}$ from S_q^n, where $\overline{0} = (0, \ldots, 0)$ is the all-zero vector of length n. If indeed $P(\overline{a}) \neq 0$, consider the following univariate algebraic circuit:

$$G(\mathrm{x}) := P(c_1, \ldots, c_{k-1}, \ \mathrm{x} \ , a_{k+1}, \ldots, a_n). \tag{30}$$

The algorithm loops through r in S_q to check if there are at least i different $r \in S_q$ such that $P(r) = 0$. If it finds less than i roots in S_q, then something went wrong and again the algorithm outputs $\overline{0} = (0, \ldots, 0)$ in S_q^n as a default value. Otherwise, it sets r to the i-th smallest root of $G(\mathrm{x})$ in S and outputs $(c_1, \ldots, c_{k-1}, r, c_k, \ldots, c_{n-1})$ in S_q^n.

Correctness of the encoding scheme. We are ready to prove that the encoding scheme for roots of P given by the pair of functions $\mathrm{enc}_{P,\overline{a}}$ and $\mathrm{dec}_{P,\overline{a}}$ is correct. This correctness hinges on the condition $P(\overline{a}) \neq 0$ but does not require any upper bound on the components of the vector $\overline{a}$. Note also that the functions $\mathrm{enc}_{P,\overline{a}}$ and $\mathrm{dec}_{P,\overline{a}}$ are polynomial-time computable because they are given $P, d, q, \overline{a}$ in the input, with d and q in unary notation. Finally, it is also worth pointing out that the correctness of the encoding scheme will not require any lower bound on q. However, as noted earlier, the encoding achieves some compression only if $nd < q$.

In the statement of the next lemma, which states the correctness, recall that we use the notation $P \equiv 0$ to the denote the (unbounded) formula $\forall \overline{a} \in \mathbb{Z}^n \; P(\overline{a}) = 0$, where $\mathbb{Z}$ denotes the set of encodings of *all* integers.

Theorem 3.9 (Proof of Schwartz-Zippel Lemma in S_2^1)**.** *For all $n, d, q \in \mathrm{Log}$ and all $P \in \mathrm{Ckt}(n, d)$, either $P \equiv 0$ or for every $\overline{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ such that $P(\overline{a}) \neq 0$ the function $\mathrm{dec}_{P,\overline{a}}$ is surjective onto $Z_{P,q}$ and inverts $\mathrm{enc}_{P,\overline{a}}$ on $Z_{P,q}$: for every $\overline{b} \in Z_{P,q}$ we have $\mathrm{dec}_{P,\overline{a}}(\mathrm{enc}_{P,\overline{a}}(\overline{b})) = \overline{b}$.*

Proof. Since for this proof the parameters $n, d, q, P, \overline{a}$ will be fixed, we drop them from the notation and write S and Z instead of S_q and $Z_{P,q}$, and dec and enc instead of $\mathrm{dec}_{P,\overline{a}}$ and $\mathrm{enc}_{P,\overline{a}}$. Given a root $\overline{b} = (b_1, \ldots, b_n) \in Z$, our goal is to find a triple $(k, i, \overline{c}) \in [n] \times [d] \times S^{n-1}$ such that $\mathrm{dec}(k, i, \overline{c}) = \overline{b}$; indeed we will show that the choice $(k, i, \overline{c}) = \mathrm{enc}(\overline{b})$ works, and this proves also the second part of the lemma that $\mathrm{dec}(\mathrm{enc}(\overline{b})) = \overline{b}$.

Fix $(k, i, \overline{c}) = \mathrm{enc}(\overline{b})$. To see that $\mathrm{dec}(k, i, \overline{c}) = \overline{b}$, recall that in the definition of $\mathrm{enc}(\overline{b})$ the index k is defined as the smallest index in the sequence $1, \ldots, n$ such (28) holds. In particular, the univariate circuit $Q(\mathrm{x})$ from (29) does not vanish everywhere as it is non-zero on a_k. Further, $Q(\mathrm{x})$ has syntactic degree at most d since it is a restriction of P with plugged constants from $\overline{a}$. Using the PV-function of Lemma 3.8, we can extract the coefficients of a polynomial $H = \mathrm{polynomial}(Q, d)$ in $\mathrm{UniPoly}(d)$ such that $H \equiv Q$; i.e., the equality $H(u) = Q(u)$ holds for all integers $u \in \mathbb{Z}$. In particular $H(a_k) = Q(a_k) \neq 0$ and, by Lemma 3.4, the function $h_{H,q}$ in surjective onto the set of roots of $H(\mathrm{x})$ in S. Also b_k is one of these roots by (28) and the just mentioned fact that $H \equiv Q$.

Now recall that in the definition of the function $\mathrm{enc}(\overline{b})$, the components i and $\overline{c}$ in its output are defined to satisfy the following conditions:

C1. element $b_k \in S$ is the i-th smallest root of $Q(\mathrm{x})$ in S *unless* $i > d$,
C2. vector $\overline{c} = (c_1, \ldots, c_{n-1})$ is set to $(b_1, \ldots, b_{k-1}, b_{k+1}, \ldots, b_n)$.

The i-th smallest root of $Q(\mathrm{x})$ in S is, by virtue of $H \equiv Q$, also the i-th smallest root of $H(\mathrm{x})$ in S. Thus, by Lemma 3.4 and the fact that $H(b_k) = Q(b_k) = 0$, this is the

preimage of b_k under $h_{H,q}$, which is at most d since the domain of $h_{H,q}$ is $[d]$. This means that, in C1, the index i is indeed such that b_k is the i-th smallest root of $Q(\mathrm{x})$ in S. We show that $\mathrm{dec}(k, i, \overline{c}) = \overline{b}$.

Let r be the i-th smallest root in S of the univariate polynomial $G(\mathrm{x})$ from (30), which exists because by C2 we have $G \equiv Q$, even syntactically as circuits in this case. By the discussion in the previous paragraph b_k is the i-th smallest root of $Q(\mathrm{x})$ in S. By the definition of the function $\mathrm{dec}(k, i, \overline{c})$ we have $\mathrm{dec}(k, i, \overline{c}) = (c_1, \ldots, c_{k-1}, r, c_k, \ldots, c_{n-1})$, which by C1 and C2 equals $(b_1, \ldots, b_{k-1}, b_k, b_{k+1}, \ldots, b_n)$, also known as $\overline{b}$. This completes the proof. □

4 Hitting Sets and Identity Testing in the Theory

4.1 Hitting Sets

So far we argued that the theory S^1_2 proves that every n-variable algebraic circuit with integer coefficients and small degree has relatively few roots in S^n_q, unless the polynomial vanishes everywhere. By a standard counting argument, it follows that for any given bounds d and m on the degree and the description size of algebraic circuits with n variables, there is a set $(\overline{c}_1, \ldots, \overline{c}_r)$ of $r = \mathsf{poly}(n, d, m)$ points in S^n_q, with $q = \mathsf{poly}(n, d)$, that intersects the set of non-roots of every non-vanishing algebraic circuit with those bounds. We say that $(\overline{c}_1, \ldots, \overline{c}_r)$ is a *hitting set over* S_q. Formally:

Definition 4.1 (In S^1_2)**.** Let $\mathscr{C}$ be a definable class of algebraic circuits (Definition 3.1). Let e be a parameter, let $n, d, s, q, r, m \in \mathrm{Log}$ be positive lengths and let $H = (\overline{h}_1, \ldots, \overline{h}_r) \in (S^n_q)^r$ be an r-sequence of n-vectors with entries in S_q. We say that H is a *hitting set for* $\mathscr{C}_e(n, d, s, m)$ *over* S_q if and only if for every $P \in \mathscr{C}(n, d, s, m)$, if there exists $\overline{a} \in \mathbb{Z}^n$ such that $P(\overline{a}) \neq 0$, then there exists $i \in [r]$ such that $P(\overline{h}_i) \neq 0$.

If H as above is not a hitting set, then a *witness* that H is not a hitting set is a pair

$$(P, \overline{a}) \in \mathscr{C}_e(n, d, s, m) \times \mathbb{Z}^n \tag{31}$$

such that $P(\overline{a}) \neq 0$ and $P(\overline{h}_j) = 0$ for every $j \in [r]$. The *size* of the hitting set H is the number of distinct tuples among the $\overline{h}_i$, which is always at most r. In what follows we use the results of the previous sections to show that the counting argument that proves that polynomial-size hitting sets exist can be formalised in the theory $\mathsf{S}^1_2 + \mathsf{dWPHP}(\mathsf{PV})$.

Small witnesses. We start by showing that, if q is sufficiently big, but polynomial in n and d, then the witnesses of failure as in (31) can always be chosen with $\overline{a}$ in S^n_q. This is interesting in its own right:

Lemma 4.2 (Small Witnesses Exist in $\mathsf{S}^1_2 + \mathsf{dWPHP}(\mathsf{PV})$)**.** *For all $n, d, q \in \mathrm{Log}$ and every algebraic circuit $P \in \mathrm{Ckt}(n, d)$, if $P \not\equiv 0$ and $q \geq 2dn$, then there exists $\overline{a} \in S^n_q$ such that $P(\overline{a}) \neq 0$.*

Proof. Assume $P \not\equiv 0$, so there exists $\overline{a} \in \mathbb{Z}^n$ such that $P(\overline{a}) \neq 0$. We find an $\overline{a}_0 \in S_q^n$ also with $P(\overline{a}_0) \neq 0$. By Theorem 3.9, the function $\mathrm{dec}_{P,\overline{a}} : C_{n,d,q} \to S_q^n$ is surjective onto $Z_{P,q}$. Let $b = q^n$ and note that the parameters q, n allow us to code each point $\overline{a}_0 \in S_q^n$ as an n-tuple of elements in $\{0, \ldots, q-1\}$. Thus, the co-domain S_q^n of $\mathrm{dec}_{P,\overline{a}}$ can be identified with the set $[q^n]$. Similarly, the parameters n, d, q allow us to identify the domain $C_{n,d,q}$ with the set $[dnq^{n-1}]$. Thus, the function $\mathrm{dec}_{P,\overline{a}}$ can be thought of as the parameterization of a PV-function f such that:

$$f_{P,\overline{a},n,d,q} : [a] \to [b] \quad \text{where} \quad \begin{array}{rcl} a & := & dnq^{n-1}, \\ b & := & q^n\,. \end{array} \tag{32}$$

Now note that the assumption $q \geq 2dn$ implies $b \geq 2a \geq 2$. Since each element of $[b]$ codes an element of S_q^n, and each element of $C_{n,d,q}$ is coded by an element of $[a]$, by $\mathsf{dWPHP}_b^a(f_{P,\overline{a},n,d,q})$, there exists $\overline{a}_0 \in S_q^n$ that is outside the range $Z_{P,q}$ of $\mathrm{dec}_{P,\overline{a}}$. Since $Z_{P,q}$ is the set of all roots of P in S_q^n, we get that $\overline{a}_0$ is a non-root of P that belongs to S_q^n, as needed. □

As stated earlier, one consequence of Lemma 4.2 is that whenever $H \subseteq S_q^n$ fails to be a hitting set in the sense of Definition 4.1, if $q \geq 2dn$ then not only is there a witness of this failure as in (31), but there is even a witness of the following form:

$$(P, \overline{a}) \in \mathscr{C}(n, d, s, m) \times S_q^n \tag{33}$$

Next we show how to use this to get hitting sets with one more application of dWPHP.

Hitting sets exist. Let $\mathscr{C}$ be a definable class of algebraic circuits, let e be a parameter, and let n, s, d, q, r, m be positive integers with $q > d$ and $s \geq n$. Consider the function

$$g_{\mathscr{C},e,n,d,s,q,r,m} : \mathscr{C}_e(n, d, s, m) \times S_q^n \times ([n] \times [d] \times S_q^{n-1})^r \to (S_q^n)^r \tag{34}$$

defined by

$$(P, \overline{a}, \overline{c}_1, \ldots, \overline{c}_r) \mapsto (\mathrm{dec}_{P,\overline{a}}(\overline{c}_1), \ldots, \mathrm{dec}_{P,\overline{a}}(\overline{c}_r)), \tag{35}$$

where $\overline{c}_1, \ldots, \overline{c}_r \in [n] \times [d] \times S_q^{n-1}$ are candidate codes for roots of P in S_q^n. Recall that if $P(\overline{a}) = 0$, then the definition of $\mathrm{dec}_{P,\overline{a}}(\overline{c})$ is such that its output is $\overline{0}$, so that $g(P, \overline{a}) = (\overline{0}, \ldots, \overline{0})$ in this case. Here, the all-zero tuple $(\overline{0}, \ldots, \overline{0})$ is used as a *don't care* value indicating that $\overline{a}$ is not serving the purpose of certifying that P does not vanish everywhere.

Lemma 4.3 (In $\mathsf{S}_2^1 + \mathsf{dWPHP}(\mathsf{PV})$)**.** *For every definable class $\mathscr{C}$ of algebraic circuits, every parameter e for its decoding function, all $n, d, s, q, r, m \in \mathrm{Log}$, and every $H \in (S_q^n)^r$, if $q \geq 2dn$ and $H \notin \mathrm{Img}(g)$, where $g = g_{\mathscr{C},e,n,d,s,q,r,m}$, then H is a hitting set for $\mathscr{C}_e(n, d, s, m)$ over S_q.*

Proof. We prove the contrapositive statement. Assume that $H = (\overline{d}_1, \ldots, \overline{d}_r)$ is not a hitting set and let $(P, \overline{a})$ witness it as in (31): $\overline{a} \in \mathbb{Z}^n \setminus Z_{P,q}$ and $\overline{d}_j \in Z_{P,q}$ for every $j \in [r]$. Since $q \geq 2dn$, by Lemma 4.2 we may assume that $\overline{a}$ is in S_q^n; i.e., the witness $(P, \overline{a})$ is as in (33): $\overline{a} \in S_q^n \setminus Z_{P,q}$ and $\overline{d}_j \in Z_{P,q}$ for every $j \in [r]$. By Lemma 3.9, there exists $(\overline{c}_1, \ldots, \overline{c}_r) \in ([n] \times [d] \times S_q^{n-1})^r$ such that $\mathrm{dec}_{P,\overline{a}}(\overline{c}_j) = \overline{d}_j$ for every $j \in [r]$. Hence, $g(P, \overline{a}, \overline{c}_1, \ldots, \overline{c}_r) = (\overline{d}_1, \ldots, \overline{d}_r)$ by the definition of g in (35). This shows that H is in the range of g. □

We intend to apply $\mathsf{dWPHP}(g)$ for the function g in (34). To do so we need to view g as a function from $[a]$ to $[b]$ for appropriate a and b such that $b \geq 2a \geq 2$. In other words, we need to specify the numeric encodings for the elements in the domain and the co-domain of g. For the co-domain, the parameters n, q, r allow us to code each element $H \in (S_q^n)^r$ as an nr-tuple of elements in $\{0, \ldots, q-1\}$, so the co-domain can be identified with the set $[q^{nr}]$. Recall that the parameters n, r are lengths, i.e., elements of Log, so q^{nr} exists. For the domain, the parameters e, n, d, s, q, r, m allow us to code each element $(P, \overline{a}, \overline{c}_1, \ldots, \overline{c}_r) \in \mathscr{C}_e(n, d, s, m) \times S_q^n \times ([n] \times [d] \times S_q^{n-1})^r$ as a number in $[2^m n^r d^r q^{(n-1)r+n}]$. Again this quantity exists because the parameters n, r, m are lengths; i.e., elements of Log. Summarizing, the specification (34) becomes

$$g_{\mathscr{C},e,n,d,s,q,r,m} : [a] \to [b] \quad \text{where} \quad \begin{array}{rcl} a & := & 2^m n^r d^r q^{(n-1)r+n}. \\ b & := & q^{nr}. \end{array} \tag{36}$$

The fact that every element in $[b]$ denotes an element of the co-domain will be used in the proof of the next theorem.

Theorem 4.4 (Small Hitting Sets Exist in $\mathsf{S}_2^1 + \mathsf{dWPHP}(\mathsf{PV})$)**.** *For every definable class $\mathscr{C}$ of algebraic circuits, every parameter e for its decoding function, and all $n, d, s, q, r, m \in \mathrm{Log}$ such that $q \geq 2dn$ and $r > m + n|q|$, there exists $H \in (S_q^n)^r$ such that H is a hitting set for $\mathscr{C}_e(n, d, s, m)$ over S_q.*

Proof. To show the existence of $H = (\overline{d}_1, \ldots, \overline{d}_r)$ as claimed in the lemma we apply the dual weak pigeonhole principle $\mathsf{dWPHP}(g)$ on the PV-function g in (34), with the parameters indicated there. By the discussion preceding the lemma we have $g : [a] \to [b]$ where a and b are as in (36). Comparing a to b, we see that the assumptions $q \geq 2nd$ and $r > m + n|q|$ yield $b \geq 2a$. Indeed, $m, n, |q|$ are integers, so $r \geq m + 1 + n|q|$. Therefore, using $q \geq 2nd$, we get $(q/(nd))^r \geq 2^{m+1+n|q|} \geq 2^{m+1} q^n$. Multiplying both sides by q^{nr} and rearranging we get $b \geq 2a$. Recalling that each element of $[b]$ codes an element of $(S_q^n)^r$, by $\mathsf{dWPHP}_b^a(g)$ there exists $H \in (S_q^n)^r$ that is outside the range of g. By Lemma 4.3, this H is a hitting set for $\mathscr{C}_e(n, d, s, m)$ over S_q, as needed. □

4.2 PIT in co-RP and in P/poly

Let *PIT* stand for *Polynomial Identity Testing*, which is the following computational problem:

> PIT: *Given positive integers n, d, s written in unary notation and given an algebraic circuit C of representation size s, with n variables, integer constants, and syntactic individual degree d, does it compute the identically zero polynomial?*

The requirement to have d written in unary notation in the input is a way to enforce that the polynomial computed by the circuit has *polynomial degree*; the size of the input is inflated at the same rate as the degree. Similarly, since our definition of degree includes the parametric

inputs, the size of the input is inflated at the same rate as the bit-complexity of the constants it computes.

In the statement of the problem, the phrase "identically zero polynomial" can be interpreted in two ways: syntactically and semantically. In the syntactic interpretation, it is the polynomial which has zero coefficient on each monomial when written explicitly in its unique representation as a finite linear combination of monomials. In the semantic interpretation, it is the polynomial that evaluates to zero on all points in $\mathbb{Z}^n$. It is well known that both interpretations yield the same computational problem, but note that weak theories such as S^1_2 or even $\mathsf{S}^1_2 + \mathsf{dWPHP}(\mathsf{PV})$ may not be able to prove this since the standard proof involves exponential-time computations.

The Schwartz-Zippel Lemma as stated in Theorem 1.1 implies that the semantic interpretation of PIT is not only decidable but even in co-NP: it suffices to check that C evaluates to zero on all points $\overline{a} = (a_1, \ldots, a_n)$ of the cube $\{0, \ldots, q-1\}^n$, where $q = 2nd$. For the sake of formalisation in weak theories, let us state what we call the *explicitly* co-NP *formulation* of the problem:

> Bounded PIT: *Given positive integers n, d, s written in unary notation and given an algebraic circuit C of representation size s, with n variables, integer constants, and syntactic individual degree d, does it evaluate to 0 on all points $(a_1, \ldots, a_n)$ of the cube $\{0, \ldots, q-1\}^n$ of side-length $q = 2nd$?*

This version of the problem is in co-NP yet the Schwartz-Zippel Lemma shows that it is even in co-RP and hence in P/poly. Indeed, if $P(\overline{a}) \neq 0$ for at least one point $\overline{a} \in \{0, \ldots, q-1\}^n$, then $P(\overline{a}) \neq 0$ for a fraction of at least $1 - nd/q \geq 1/2$ of the points $\overline{a} \in \{0, \ldots, q-1\}^n$. This means that an evaluation at a random point in $\{0, \ldots, q-1\}^n$ has chance at least $1/2$ of detecting that C does not always evaluate to zero. This is co-RP. The results in this paper show that these statements are available in the theory $\mathsf{S}^1_2 + \mathsf{dWPHP}(\mathsf{PV})$. For P/poly the definition of what this means is straightforward. Let Boolean circuits be encoded in the theory in a completely analogous way to how algebraic circuits are encoded in the theory. The only difference is that the evaluation function for Boolean circuits takes only binary strings as inputs for its variables or parameters, and interprets its gates in the Boolean sense, with $+$ as disjunction, and $\times$ as conjunction.

Definition 4.5 (In S^1_2)**.** Let $c \geq 1$ be a standard constant and let A be a definable set of strings given by a formula $\varphi(x)$ without parameters, i.e., with all free variables indicated. The set A is in $\mathsf{SIZE}(n^c)$ if for every $n \in \mathrm{Log}$ there exists a Boolean circuit C with n^c gates, with n inputs and 1 output, such that for every $x \in \{0,1\}^n$ we have that $C(x) = 1$ if and only if $\varphi(x)$ holds.

Membership of PIT in P/poly is now an immediate consequence of the existence of hitting sets. We state this explicitly. Let PIT denote the Π_1-definable set of strings that encode the YES instances of the computational problem defined earlier: the strings encode 4-tuples (C, n, d, s) where n, d, s are integers presented in unary notation (i.e., as three strings of the form $100\cdots0$ with lengths n, d, s), and C is an algebraic circuit in $\mathrm{Ckt}(n, d, s)$

such that $C \equiv 0$ holds; i.e., the Π_1-formula $\forall \overline{a} \in \mathbb{Z}^n\ C(\overline{a})=0$ holds. Similarly, let Bounded PIT denote the Π^b_1-definable set of strings that encode the YES instances of the explicitly co-NP version of PIT. We say that a theory proves that a definable set A of strings is in P/poly if there exists a standard constant $c > 0$ such that it proves that A is in $\mathsf{SIZE}(n^c)$.

Theorem 4.6 (PIT is in P/poly in $\mathsf{S}^1_2 + \mathsf{dWPHP}(\mathsf{PV})$)**.** *The theory $\mathsf{S}^1_2 + \mathsf{dWPHP}(\mathsf{PV})$ proves that both PIT and Bounded PIT are in P/poly. Indeed, there is a standard constant $c \geq 1$, such that it proves the following stronger statement proving both claims simultaneously: For all $n, s, d, q \in \mathrm{Log}$ such that $q \geq 2nd$, there exists a Boolean circuit C with $(n + d + s + q)^c$ gates such that for every $P \in \mathrm{Ckt}(n, d, s)$ the following hold:*

1. *if $C(P) = 1$, then for every $\overline{a} \in \mathbb{Z}^n$ we have $P(\overline{a}) = 0$,*
2. *if $C(P) = 0$, then there exists $\overline{a} \in S^n_q$ such that $P(\overline{a}) \neq 0$.*

Proof. Consider the PV function that given (C, n, d, s) as required in the input for PIT and given $H = (\overline{h}_1, \ldots, \overline{h}_r) \in S^n_q$, evaluates C on all points in H and outputs 1 if all evaluations are zero, and outputs 0 otherwise. Setting $q = 2nd$ and $r = s + n|q|$, and hardwiring for H the hitting set for $\mathrm{Ckt}(n, d, s)$ of Theorem 4.4 we get a Boolean circuit C as required, for a standard $c \geq 1$ that depends only on the runtime of the PV function. □

Corollary 4.7 (Small Counterexamples for Identities in $\mathsf{S}^1_2 + \mathsf{dWPHP}(\mathsf{PV})$)**.** *The theory $\mathsf{S}^1_2 + \mathsf{dWPHP}(\mathsf{PV})$ proves that PIT and the explicitly co-NP version of PIT are the same problem, i.e., the formulas that define the sets PIT and Bounded PIT are equivalent.*

Stating membership of PIT in co-RP within the theory $\mathsf{S}^1_2 + \mathsf{dWPHP}(\mathsf{PV})$ is much less straightforward, but at least the necessary ingredients are now available. We refer the reader to Jeřábek's seminal papers [20, 21] where the complexity classes BPP and RP are studied in $\mathsf{PV}_1 + \mathsf{dWPHP}(\mathsf{PV})$, a subtheory of $\mathsf{S}^1_2 + \mathsf{dWPHP}(\mathsf{PV})$.

5 Complexity of Finding Hitting Sets

5.1 Hitting Set Axioms

Let $\mathscr{C}$ be a definable class of algebraic circuits with parameterized slices $\mathscr{C}_e(n, d, s, m)$ according to the definition. To recall, the slice $\mathscr{C}_e(n, d, s, m)$ is the set of algebraic circuits in the class $\mathscr{C}$ with parameter e that have at most n indeterminates, syntactic individual degree at most d, representation size at most s, but description size $m \leq s$ as members of $\mathscr{C}$. Let the *Hitting Set Axiom for the class $\mathscr{C}$* be the following statement:

$$\begin{aligned}\mathsf{HS}(\mathscr{C}) \ :=\ & \forall e\ \forall n, d, s, q, r, m \in \mathrm{Log}\ (q \geq 2nd \wedge r > m + n|q| \rightarrow \exists H = (\overline{h}_1, \ldots, \overline{h}_r) \in (S^n_q)^r \\ & \forall C \in \mathscr{C}_e(n, d, s, m)\ ((\exists \overline{a} \in \mathbb{Z}^n\ C(\overline{a}) \neq 0) \rightarrow (\exists i < r\ C(\overline{h}_{i+1}) \neq 0))).\end{aligned}$$

It is important to note that the *largeness* hypothesis $r > m + n|q|$ does *not* depend on the representation size s; it depends only on the description size m, the bit-complexity $|q|$ of the

evaluation points, and on the number n of indeterminates. The role of the representation size s in $\mathscr{C}_e(n,d,s,m)$ is to ensure that the circuits of the class $\mathscr{C}$ can be evaluated in polynomial time, given n,d,s in unary, a description $x \in \{0,1\}^m$, and an integer assignment for the variables. When the definable class $\mathscr{C}$ is simply the class Ckt of all circuits, then $m = s$ and the largeness bound is $r > s + n|q|$.

We write $\mathsf{HS}(\mathsf{PV})$ for the axiom-scheme formed by all axioms of the form $\mathsf{HS}(\mathscr{C})$ where $\mathscr{C}$ is a definable class of algebraic circuits. To motivate this notation, we observe that if g is the encoding function of a definable class $\mathscr{C}$, then the axiom $\mathsf{HS}(\mathscr{C})$ would be provably equivalent (in S^1_2) to the following slightly longer but essentially identical variant:

$$\begin{aligned}\mathsf{HS}(g) \;:=\; \forall e\,\forall n,d,s,q,r,m{\in}\mathrm{Log}\;(q{\geq}2nd \wedge r{>}m{+}n|q| \rightarrow \exists H{=}(\overline{h}_1,\ldots,\overline{h}_r){\in}(S^n_q)^r\\ \forall x{\in}\{0,1\}^m\;\forall C{\in}\mathrm{Ckt}(n,d,s)\;(g_e(n,d,s,x){=}C \rightarrow\\ ((\exists \overline{a}{\in}\mathbb{Z}^n\; C(\overline{a}){\neq}0) \rightarrow (\exists i{<}r\; C(\overline{h}_{i+1}){\neq}0)))).\end{aligned}$$

Conversely, every axiom $\mathsf{HS}(g)$ with g in PV is provably equivalent (in S^1_2) to the axiom $\mathsf{HS}(\mathscr{C})$ for the definable class $\mathscr{C}$ that has as encoding function the modification of g, still in PV, that with parameter e and input (n,d,s,x) outputs a default trivial circuit that computes the constant 0 polynomial if $g_e(n,d,s,x)$ is not already a circuit in $\mathrm{Ckt}(n,d,s)$.

Let us make turn this notation into a definition:

Definition 5.1. For every PV-symbol g, the *hitting set axiom for* g is the formula $\mathsf{HS}(g)$. The *hitting set axiom scheme*, denoted by $\mathsf{HS}(\mathsf{PV})$, is the collection of all $\mathsf{HS}(g)$ for all $g \in \mathsf{PV}$.

An immediate consequence of Theorem 4.4 is that $\mathsf{HS}(\mathsf{PV})$ is a consequence of $\mathsf{dWPHP}(\mathsf{PV})$ over S^1_2. The first goal of this section is to prove the converse:

Theorem 5.2 (Reverse Mathematics of Hitting Sets)**.** *The axiom-schemes* $\mathsf{dWPHP}(\mathsf{PV})$ *and* $\mathsf{HS}(\mathsf{PV})$ *are provably equivalent over* S^1_2.

The next two sections are devoted to the proof of Theorem 5.2.

That $\mathsf{S}^1_2 + \mathsf{dWPHP}(\mathsf{PV})$ proves $\mathsf{HS}(\mathsf{PV})$ was shown in Theorem 4.4. Here we show the converse. Fix a PV-symbol f for which we want to prove $\mathsf{dWPHP}(f)$. Let a and b be such that $b \geq 2a \geq 2$, and let c be a setting for the parameters of f. Our goal is to show that there exists $y \in [b]$ such that for every $x \in [a]$ we have $f_c(x) \neq y$.

The first step in the proof is to apply an amplification transformation to f_c that, without loss of generality, will let us assume that $a = 2^m$ and $b = 2^{m+t}$, for $m = |a-1|$ and a $t = \mathsf{poly}(m)$ of our choice. This will let us think of f_c as a function $h_e : \{0,1\}^m \rightarrow \{0,1\}^{m+t}$. The goal will then become finding a string $y \in \{0,1\}^{m+t}$ that is outside the range of the function h_e.

The amplification technique that we need is standard in the context of the weak pigeonhole principle. For the sake of completeness, the details of this argument have been made explicit in Section A of the Appendix. Concretely, we summarize the composition of Lemmas A.1 and A.2 in the following lemma. In its statement, PV_2 refers to a collection of symbols for the (clocked) $\mathsf{FP}^{\mathsf{NP}}$-machines, just as PV is a collection of symbols

for (clocked) FP-machines. Indeed, we need symbols for the $\mathsf{FP}^{\mathsf{NP}}[\mathsf{wit}, q]$-machines: $\mathsf{FP}^{\mathsf{NP}}$-machines that get witnesses to their NP-oracle queries when they are answered YES, and make at most q queries in every computation path. In general, such machines compute multi-output functions (i.e., total relations), instead of just functions, because their output may depend on the witness-sequence provided by the oracle. The $\mathsf{FP}^{\mathsf{NP}}[\mathsf{wit}, O(\log n)]$-computable multi-output functions are Σ^b_2-definable in S^1_2 by Theorem 6.3.3 in [26].

Lemma 5.3 (In S^1_2)**.** *For every PV-function f and every standard natural number $k \geq 2$, there is a PV-function h and a $\mathsf{PV}_2[\mathsf{wit}, 2]$-function r such that for all a, b, c such that $b \geq 2a \geq 2$, setting $m = |a - 1|$ and $e = \langle a, c\rangle$ as parameter for the functions h and r, the restriction of h_e to $\{0,1\}^m$ is a function $h_e : \{0,1\}^m \to \{0,1\}^{m^k}$ and, for every $y \in \{0,1\}^{m^k}$ that is outside the range of h_e, there is a computation of $r_e(y)$ that does not fail, and any such computation outputs an element in $[2a] \subseteq [b]$ that is outside the range of f_c restricted to $[a]$.*

Proof. Set $t = m^k - m$ and compose the g and h functions, and the s and r functions, in Lemmas A.1 and A.2 of Section A. □

The bottom line of this section is that Lemma 5.3 reduces the problem of producing a witness for $\mathsf{dWPHP}^a_b(f_c)$ to the problem of producing a string in $\{0,1\}^{m^k}$ that avoids the range of h_e. In our application we will choose $k := 3$, so that

$$h_e : \{0,1\}^m \to \{0,1\}^{m^3}. \tag{37}$$

The reason for the choice $k = 3$ will become clear later in the proof.

5.2 The Definable Class of Algebraic Circuits

We continue now with the proof of the implication from $\mathsf{HS}(\mathsf{PV})$ to $\mathsf{dWPHP}^a_b(f_c)$ over S^1_2. We argued already that the problem reduces to avoiding the range of h_e in (37).

For the sake of contradiction, assume that for all $y \in \{0,1\}^{m^3}$ there is an $x \in \{0,1\}^m$ such that $h_e(x) = y$. We intend to use the hitting set H provided by $\mathsf{HS}(\mathscr{C})$ for an appropriately chosen definable class of algebraic circuits $\mathscr{C}$, with suitable parameters e, n, d, s, q, r, m defined from e and m. The contradiction will come from the following three steps of reasoning:

1. first we define a string $y \in \{0,1\}^{m^3}$ that suitably encodes the hitting set H;
2. next we use the assumption to get a shorter $x \in \{0,1\}^m$ that can recover y via h_e;
3. finally, we use x, *as a compressed version of H,* to build a small non-vanishing algebraic circuit C in $\mathscr{C}_e(n, d, s, m)$ which will, however, vanish on H, by design.

Before we can execute this plan first we need to define the suitable class $\mathscr{C}$ on which to apply $\mathsf{HS}(\mathscr{C})$. We commit to the following choice of parameters e, n, d, s, q, r, m as functions of e and m. First we note that we are not overloading the names of the parameters: e and m will be taken to be themselves. Second, since every finite instance of dWPHP of standard

size is provable in BASIC, and in particular in S^1_2, we may assume that m is large enough, i.e., $m \geq m_0$ for a fixed standard m_0 to be fixed later. For such large m, choose

$$\begin{array}{lll} n := m & d := m^2 & s := m^3 \\ r := 4m|m| & q := 2m^3. & \end{array} \tag{38}$$

Observe that, if m is large enough, then

$$m^3 \geq rn|q| \qquad d \geq 2r|q| \qquad q \geq 2dn \qquad r > m + n|q|. \tag{39}$$

The first part in (39) implies that every triple $(i, j, k) \in [r] \times [n] \times [|q|]$ can be encoded with a number in $[m^3]$.[2] Fix such an encoding of triples and write $e_m(i, j, k) \in [m^3]$ for the encoding of (i, j, k); we require of this encoding that it is provably PV-computable and provably PV-invertible, which is easy to achieve in many standard ways, given m as parameter.

We now define the class $\mathscr{C}$. For all $x \in \{0,1\}^m$ and $(i, j, k) \in [r]\times[n]\times[|q|]$, let $h_e(x; i, j, k)$ denote the $e_m(i, j, k)$-th bit of output of $h_e(x)$. Let $\mathscr{G}_{e,m} \subseteq \mathrm{Ckt}(n, d)$ be the set of algebraic circuits with variables $z_1, \ldots, z_n$ and plugged constants $-1, 0, 1$ that have the form

$$A_{e,x}(z_1, \ldots, z_n) := \prod_{i\in[r]} \sum_{j\in[n]} \Big(z_j - \sum_{k\in[|q|]} h_e(x; i, j, k) \cdot 2^{k-1}\Big)^2, \tag{40}$$

where $x \in \{0,1\}^m$. In this description of the algebraic circuit $A_{e,x}$, the $z_1, \ldots, z_n$ are the variables, while the constants $-1, 0, 1$ are used for the $h_e(x; i, j, k)$, the 2 in 2^{k-1}, and the -1 that is implicit in the subtraction. The syntactic individual degree of (40) on the z_j variables is $2r$, and the syntactic individual degree of (40) on the parametric inputs is at most $2r|q|$. By the second part in (39) we have $d \geq 2r|q|$, so indeed, as claimed, $\mathscr{G}_{e,m}$ is a subset of $\mathrm{Ckt}(n, d)$. Further, if m is large enough, by the first part in (39) again, the representation size of $A_{e,x}$ (as a member of Ckt) is bounded by m^3. Thus, by the choice of s in (38), the set $\mathscr{G}_{e,m}$ is even a subset of $\mathrm{Ckt}(n, d, s)$. We let $\mathscr{C}$ be given by the encoding function $g_e(n, d, s, x) = A_{e,x}$, if n, d, s, m obey (38) and m is as determined by $e = \langle a, c\rangle$, i.e., $m = |a - 1|$; otherwise we let $g_e(n, d, s, x)$ be a default trivial circuit that computes the constant 0 polynomial. By construction, $\mathscr{C}$ is a legitimate definable class of algebraic circuits and $\mathscr{C}_e(n, d, s, m) := \mathscr{G}_{e,m}$.[3]

5.3 The Compression Argument

We are ready to complete the proof of $\mathsf{dWPHP}(f)$ in $\mathsf{S}^1_2 + \mathsf{HS}(\mathsf{PV})$. Consider $\mathsf{HS}(g)$ or, equivalently, $\mathsf{HS}(\mathscr{C})$, for the class $\mathscr{C}$ of the previous section with encoding function g, and

[2] This is the place in proof where the choice $k = 3$, so h_e has co-domain $\{0,1\}^{m^3}$, is needed.

[3] Observe that $\mathscr{C}_e(n, d, s, m)$ is a *sparse* class since $2^m \ll 2^{m^3} = 2^s$. This is the crucial point that makes the proof work. We cannot afford $m = s$ as would be the case for $\mathrm{Ckt}(n, d, s)$ because the *representation size* of $A_{e,x}$ as an explicit circuit is roughly $rn|q|$, which is larger than s, and no choice of parameters can make $rn|q|$ smaller than s if $m = s$ and r and m are to satisfy the requirement that $r > m + n|q|$ in the statement of HS. In contrast, as member of $\mathscr{C}$, the *description size* of $A_{e,x}$ is m, which is significantly smaller than $rn|q| \geq m^2$.

consider the parameters e, n, d, s, q, r, m chosen in (38). These choices are made to guarantee that the hypotheses in the $\mathsf{HS}(\mathscr{C})$-axiom are met; concretely, by (39), the requirement $q \geq 2nd$ holds and, most importantly, the requirement $r > m+n|q|$, which ensures the abundance of hitting sets H, also holds.

Let then $H = (\overline{h}_1, \ldots, \overline{h}_r) \in (S_q^n)^r$ be a hitting set for $\mathscr{C}_e(n, d, s, m)$. Let $y \in \{0,1\}^{m^3}$ be the string that encodes H in binary padded with 0s to length m^3. More precisely, first note that H can be identified with an element of $[q]^{nr}$, and hence with a string in $\{0,1\}^{rn|q|}$, so the first inequality in (39) ensures that m^3 is big enough to encode H. Even more explicitly, if we write $\overline{h}_i = (h_{i,1}, \ldots, h_{i,n})$ and write each $h_{i,j} \in S_q = \{0, \ldots, q-1\}$ in binary notation as $h_{i,j} = (h_{i,j,1}, \ldots, h_{1,j,|q|}) \in \{0,1\}^{|q|}$, then we choose y such that for all triples $(i, j, k) \in [r] \times [n] \times [|q|]$ we have that

$$\text{the } e(i,j,k)\text{-th bit of } y \text{ equals } h_{i,j,k}\text{; the rest of bits of } y \text{ are 0.} \tag{41}$$

By the assumption that dWPHP fails for $h_e : \{0,1\}^m \to \{0,1\}^{m^3}$, there exists an $x \in \{0,1\}^m$ such that $h_e(x) = y$. In particular, by (41) and the definition of $h_e(x; i, j, k)$, for all $(i, j, k) \in [r] \times [n] \times [|q|]$ we have

$$h_e(x; i, j, k) = h_{i,j,k}. \tag{42}$$

Now consider the algebraic circuit $A_{e,x}(z_1, \ldots, z_n)$ from (40), which is in $\mathscr{C}_e(n, d, s, m)$ by definition. The polynomial computed by $A_{e,x}$ does not evaluate to zero on all $\overline{a} \in \mathbb{Z}^n$. To see this, take $a_j := 2q$ for $j = 1, \ldots, n$ and note that each factor of the product in $A_{e,x}(a_1, \ldots, a_n)$ is (provably) positive since each of its n summands is (provably) positive as all inner sums are less than $2^{|q|} \leq 2q$. Therefore, since H hits $A_{e,x}$, there exists $i \in [r]$ such that $A_{e,x}(\overline{h}_i) \neq 0$. But then each factor evaluated at $\overline{h}_i$ is non-zero, and in particular the i-th factor is non-zero, which means that there exists $j \in [n]$ such that

$$h_{i,j} \neq \sum_{k \in [|q|]} h_e(x; i, j, k) \cdot 2^{k-1}. \tag{43}$$

In turn, since $h_{i,j}$ belongs to $S_q = \{0, 1, \ldots, q-1\}$ and we encoded it in binary with $|q|$ bits, it follows from the (provable) uniqueness of the binary encoding that there exists $k \in [q]$ such that $h_{i,j,k} \neq h_e(x; i, j, k)$. This contradicts (42), and the claim is proved.

This completes the proof of Theorem 5.2, and this section.

5.4 Completeness for Range Avoidance Problems

Consider the total search problem of finding hitting sets for low-degree algebraic circuits:

> Hitting Sets: *Given n, d, s, q, r in unary such that $q \geq 2nd$ and $r > s + n|q|$, find a hitting set for $\mathrm{Ckt}(n, d, s)$ over S_q of size at most r.*

This is a total function in the complexity class $\mathsf{TF\Sigma_2P}$ of total functions in $\mathsf{F\Sigma_2P}$; the totality is given by Theorem 4.4. We define also its generalization to arbitrary circuit classes. In

its definition below, the phrase *a description of a class of at most 2^m algebraic circuits $\mathscr{G} \subseteq \mathrm{Ckt}(n,d,s)$* refers to a Boolean circuit C with m Boolean inputs and s Boolean outputs syntactically guaranteed to satisfy $C(x) \in \mathrm{Ckt}(n,d,s)$ for all $x \in \{0,1\}^m$. One way to fulfill the syntactic guarantee is to check its output for membership in $\mathrm{Ckt}(n,d,s)$ and to return the code of a default trivial circuit that computes the constant 0 polynomial if the check fails.

> Hitting Sets for Circuit Classes: *Given n, d, s, q, r, m in unary such that $q \geq 2nd$ and $r > m + n|q|$, and given a description of a class of at most 2^m algebraic circuits $\mathscr{G} \subseteq \mathrm{Ckt}(n,d,s)$, find a hitting set for $\mathscr{G}$ over S_q of size at most r.*

Again Theorem 4.4 shows that this is a total function in $\mathsf{TF\Sigma_2P}$. Indeed, Theorem4.4 shows that both problems reduce, in polynomial time, to the Range Avoidance Problem for Boolean circuits from [41]. In the terminology of [24, 25], this is the problem Empty, and then both problems belong to the complexity class APEPP for which Empty is its defining problem. The first of the two problems is even in the subclass SAPEPP of sparse problems in APEPP.

We observe now that the proof of Theorem 5.2 shows that the problem about hitting sets for circuit classes is complete for the class APEPP. In the precise statement below, recall that a polynomial-time mapping reduction between search problems is a pair of polynomial-time computable maps f and r: the *forward* f maps any instance x of the first problem to an instance $f(x)$ of the second problem; the *backward* r maps any solution y to the instance $g(x)$ of the second problem to a solution $h(x,y)$ of the first problem. We call this a $\mathsf{P/P}$-reduction because both maps are polynomial-time computable. If the second map is computable only by a $\mathsf{P^{NP}}$-machine, then we call it a $\mathsf{P/P^{NP}}$-reduction.

Theorem 5.4. *The problem* Hitting Sets for Circuit Classes *is complete for the* APEPP *under* $\mathsf{P^{NP}}$*-reductions; indeed,* $\mathsf{P/P^{NP}}$*-reductions suffice.*

Proof. Membership in the class follows from the proof of the first half of Theorem 5.2; i.e., from the proof of Theorem 4.4. Precisely, that proof shows that our problem reduces to Empty with a $\mathsf{P/P}$-reduction. Completeness for APEPP follows from the proof of the second half of Theorem 5.2. In more details, we argue that the proof of Theorem 5.2 shows how to reduce Empty to our problem, with a $\mathsf{P/P^{NP}}$-reduction.

Let an instance C of Empty be given: C is a Boolean circuit with m inputs and $m+1$ outputs. Let $a = 2^m$ and $b = 2^{m+1}$, let c be the integer encoding of the Boolean circuit C, and let $e = \langle a, c\rangle$. Let f be the PV-function which, with parameter c, computes as the Boolean circuit C on input $x \in \{0,1\}^m$. Let h be the PV-function given by Lemma 5.3 with $k = 3$. Thus h_e is a function as in (37).

Consider the corresponding definable class of algebraic circuits $\mathscr{C}$ from Section 5.2 and let g be its encoding function. For the given m, consider the setting of parameters n, d, s, q, r as specified in (38). By (39), these parameters satisfy the requirements in the input of our problem. By definition we have $g_e(n,d,s,x) = A_{e,x}$ for all $x \in \{0,1\}^m$; i.e., $g_e(n,d,s,x)$ outputs (the code of) the algebraic circuit $A_{e,x}$. Let D be the canonical Boolean circuit that

computes as $g_e(n,d,s,x)$ on all inputs $x \in \{0,1\}^m$. Then D is a description of the class of at most 2^m algebraic circuits $\mathscr{G}_{e,m} \subseteq \mathrm{Ckt}(n,d,s)$. The proof in Section 5.3 shows that any hitting set H of size at most r for $\mathscr{G}_{e,m}$ gives a string $y \in \{0,1\}^{m^3}$ that is outside the range of h_e restricted to $\{0,1\}^m$. Concretely, let y be the string specified in (41). Finally, let r be the $\mathsf{PV}_2[\mathsf{wit},2]$-function r from Lemma 5.3 and consider (any) $r_e(y)$. By 5.3 and the natural correspondence between $[a]$ and $\{0,1\}^m$, and between $[b]$ and $\{0,1\}^{m+1}$, this is a string in $\{0,1\}^{m+1}$ that is not in the range of C, so a solution to the Empty-instance C. Since the circuit D is computable from e and hence from C in time polynomial in the size of C, the forward map is computable by a polynomial-time machine. Since r is computable by a P^{NP}-machine and m is bounded by the size of C, the *backward* map is computable by a P^{NP}-machine. Both together give a $\mathsf{P}/\mathsf{P}^{\mathsf{NP}}$-reduction and the proof is complete. □

A Amplification for dWPHP

The material of this appendix, or at least the techniques in it, can be considered known. Since we were not able to find a reference with the exact result that we need, we provide the details here. In bounded arithmetic, the main ideas go back to the seminal paper by Paris-Wilkie-Woods [37], and have been revisited several times; cf. [31, 4] for bounded arithmetic, and [24, 25] for complexity theory. Much earlier, in the area of cryptography, exactly the same construction was used to build pseudo-random number generators (PRNGs) from hardcore bits [7], which actually give a kind of guarantee that is incomparable with what we need.

In few words, what is done here is to show that the search problem associated to an arbitrary instance of the Dual Weak Pigeonhole Principle dWPHP^a_b with $b \geq 2a$ reduces to certain special instances of the same problem. In what we call *the normalization step* below, we show how to reduce an arbitrary instance with $b \geq 2a$ to an instance of the form $a = 2^m$ and $b = 2^{m+1}$. In the terminology of Section 5.4, this will be a P/P-reduction. In *the amplification step*, also below, we show how to reduce an instance of the form $a = 2^m$ and $b = 2^{m+1}$, to an instance of the form $a = 2^m$ and $b = 2^{m+t}$ for any arbitrary $t = \mathsf{poly}(m)$ of our choice. In both cases, we do this provably in S^1_2, as this is what is needed for the proof of Theorem 5.2.

To make this section more self-contained, let us recall the definition of the Dual Weak Pigeonhole Principle axioms. Let f be a PV-symbol. The axiom $\mathsf{dWPHP}(f)$ is the universal closure of the following formula with free variables a, b, c:

$$\mathsf{dWPHP}^a_b(f_c) := (b \geq 2a \geq 2 \rightarrow \exists y \in [b]\ \forall x \in [a]\ f_c(x) \neq y). \tag{44}$$

Here, $f_c(x)$ is alternative notation for $f(x,c)$, thinking of c as a *parameter* for f. These parameters may, in particular, specify the sizes a and b of the intended domain and range of a function $f_c : [a] \to [b]$, but this is not enforced.

Let $m = |a-1|$ and $n = |b-1|$, so that $n > m$ by $b \geq 2a \geq 2^{m+1}$. In particular, all elements in $\{0,\ldots,a-1\}$ and $\{0,\ldots,b-1\}$ can be represented in binary with m and n bits,

respectively. Subtracting and adding one as appropriate, the same applies to $[a] = \{1, \ldots, a\}$ and $[b] = \{1, \ldots, b\}$.

A.1 Normalization step

In the normalization step we want to find a function $g_d : \{0,1\}^m \to \{0,1\}^{m+1}$, with its parameters d defined in terms of the given PV-function f and the parameters a, b, c, in such a way that, given a point in $\{0,1\}^{m+1}$ that is outside the range of the function g_d, it is easy to find a point in $[b]$ that is outside the range of the function f_c restricted to $[a]$. In this case, *easy* means in time polynomial in m and the length $|c| + |d|$ of the parameters.

Let us note that if a and b were both exact powers of two, then it would suffice to take $m = |a - 1|$ and let g_d compute the same as f_c with all inputs encoded in binary notation. In case a or b are not exact powers of two, we can still take $m = |a - 1|$, but the details of the construction require a bit of care to deal with the appropriate encodings of the sets. The proof is still straightforward, just more tedious.

Lemma A.1 (In S^1_2)**.** *For every* PV*-function f there exist* PV*-functions g and r such that the following holds: For all a, b, c such that $b \geq 2a \geq 2$, setting $m = |a-1|$ and $d = \langle a, b, c\rangle$, the functions g_d and r_d have domains and co-domains $g_d : \{0,1\}^m \to \{0,1\}^{m+1}$ and $r_d : \{0,1\}^{m+1} \to [2a] \subseteq [b]$, and for all $y \in \{0,1\}^{m+1}$, if y is outside the range of g_d restricted to $\{0,1\}^m$, then $r_d(y)$ is outside the range of f_c restricted to $[a]$.*

Proof. Recall that $m = |a-1|$ and let $\mathrm{num}_a : \{0,1\}^m \to [a]$ be a canonical surjection onto $[a]$ defined by, say, $\mathrm{num}_a(x_1, \ldots, x_m) = \sum_{i=1}^m x_i 2^{i-1} \bmod a$, with the residue classes mod a naturally identified with the elements of the set $[a]$. Fix a corresponding easily computable inverse function $\mathrm{num}_a^{-1} : [a] \to \{0,1\}^m$ satisfying the invertibility condition

$$\mathrm{num}_a(\mathrm{num}_a^{-1}(y)) = y \tag{45}$$

for all $y \in [a]$. Similarly, but dually, note that $|2a-1| = m+1$, so let $\mathrm{bin}_{2a} : [2a] \to \{0,1\}^{m+1}$ be a canonical surjection onto $\{0,1\}^{m+1}$, with corresponding inverse $\mathrm{bin}_{2a}^{-1} : \{0,1\}^{m+1} \to [2a]$ satisfying the invertibility condition

$$\mathrm{bin}_{2a}(\mathrm{bin}_{2a}^{-1}(\overline{y})) = \overline{y} \tag{46}$$

for all $\overline{y} \in \{0,1\}^{m+1}$. All these are PV-functions and the invertibility conditions (45) and (46) are provable in S^1_2.

Let $g_d : \{0,1\}^m \to \{0,1\}^{m+1}$ be the function defined on $\overline{x} \in \{0,1\}^m$ by

$$g_d(\overline{x}) := \mathrm{bin}_{2a}(f_c(\mathrm{num}_a(\overline{x})) \bmod 2a). \tag{47}$$

We claim that, to find a $y \in [b]$ such that $f_c(x) \neq y$ for all $x \in [a]$, it suffices to find a $\overline{y} \in \{0,1\}^{m+1}$ such that $g_d(\overline{x}) \neq \overline{y}$ for all $\overline{x} \in \{0,1\}^m$. Indeed, given such a $\overline{y} \in \{0,1\}^{m+1}$,

just let $y = r_d(\overline{y}) := \mathrm{bin}_{2a}^{-1}(\overline{y}) \in [2a] \subseteq [b]$. To prove the correctness, note that any $x \in [a]$ such that $f_c(x) = y$ would give $\overline{x} = \mathrm{num}_a^{-1}(x) \in \{0,1\}^m$ such that

$$\begin{aligned} g_c(\overline{x}) &= g_c(\mathrm{num}_a^{-1}(x)) = \mathrm{bin}_{2a}(f_c(\mathrm{num}_a(\mathrm{num}_a^{-1}(x))) \bmod 2a) = \\ &= \mathrm{bin}_{2a}(f_c(x) \bmod 2a) = \mathrm{bin}_{2a}(y \bmod 2a) = \mathrm{bin}_{2a}(y) = \\ &= \mathrm{bin}_{2a}(\mathrm{bin}_{2a}^{-1}(\overline{y})) = \overline{y}, \end{aligned}$$

where the first equality follows from the choice of $\overline{x}$, the second follows from (47), the third follows from (45) on the fact that $x \in [a]$, the fourth follows from the assumption that $f_c(x) = y$, the fifth follows from the fact that $y \in [2a]$, the sixth follows from the choice of y, and the last follows from (46) on the fact that $\overline{y} \in \{0,1\}^{m+1}$. □

A.2 Amplification step

In the amplification step we extend the co-domain of the function $g_d : \{0,1\}^m \to \{0,1\}^{m+1}$ to the set $\{0,1\}^{m+t}$, for any desired $t = \mathsf{poly}(m)$. The construction of this new function $h_d : \{0,1\}^m \to \{0,1\}^{m+t}$ will be such that, given a point outside the range of h_d, it will be possible to get a point outside the range of g_c. In this case, the sense in which it is *easy* to translate the solution is a bit more nuanced since we will need the help of an NP-oracle in the computation. Concretely, the translation function will be represented by a PV_2-symbol that computes a multi-output function (i.e., a total relation) in $\mathsf{FP}^{\mathsf{NP}}[\mathsf{wit}, 2]$. Here, $\mathsf{FP}^{\mathsf{NP}}[\mathsf{wit}, q]$ denotes the class of multi-output functions that can be computed by $\mathsf{FP}^{\mathsf{NP}}$-machines that get witnesses to their NP-oracle queries when they are answered YES, and make at most q queries in every computation path. For $q = O(\log n)$, where n is the size of the input, the computations of such machines are Σ_2^b-definable in S_2^1 by Theorem 6.3.3 in [26].

Lemma A.2 (In S_2^1)**.** *For every PV-function g there exist a PV-function h and a $\mathsf{PV}_2[\mathsf{wit}, 2]$-function s such that the following holds: For all d and all $m, t \in \mathrm{Log}$, setting $e = \langle d, m, t\rangle$ we have that h_e and s_e compute functions $h_e : \{0,1\}^m \to \{0,1\}^{m+t}$ and (multi-output) $s_e : \{0,1\}^{m+t} \to \{0,1\}^{m+1}$, and for all $y \in \{0,1\}^{m+t}$, if y is outside the range of h_e restricted to $\{0,1\}^m$, then there is a computation of $s_e(y)$ that does not fail, and any such computation outputs a string outside the range of g_d restricted to $\{0,1\}^m$.*

We begin the proof of Lemma A.2 by fixing notation to manipulate strings. For a string $x = (a_1, \ldots, a_k) \in \{0,1\}^k$ and indices $i, j \in [k]$ we write $x[i,j]$ to denote the substring $(a_i, a_{i+1}, \ldots, a_j)$ of x between positions i and j with endpoints included (unless $i > j$). This is a string in $\{0,1\}^{j-i+1}$ if $i \leq j$, and the empty string if $i > j$. For strings $x = (a_1, \ldots, a_k) \in \{0,1\}^k$ and $y = (b_1, \ldots, b_\ell) \in \{0,1\}^\ell$, we write $x{:}y$ to denote the concatenation $(a_1, \ldots, a_k, b_1, \ldots, b_\ell)$ of x and y, which is a string in $\{0,1\}^{k+\ell}$.

With this notation, we define the function h_e. For $i = 0, 1, \ldots, t$, let $h_{e,i} : \{0,1\}^m \to \{0,1\}^{m+i}$ be defined for all $x \in \{0,1\}^m$ by the following recursion:

$$\begin{aligned} h_{e,0}(x) &= x \\ h_{e,1}(x) &= g(x) \\ h_{e,i+2}(x) &= g_d(h_{e,i+1}(x)[1,m]){:}h_{e,i+1}(x)[m+1, m+i+1]. \end{aligned}$$

Note that for any $t = \mathsf{poly}(m)$, the function $h_{e,i} : \{0,1\}^m \to \{0,1\}^{m+t}$ is computable in polynomial time. Let h be the corresponding PV-symbol in such a way that its parameterization by e, i is $h_{e,i}$, and its parameterization by e alone is $h_{e,t}$. We claim that, to find a $y_0 \in \{0,1\}^{m+1}$ such that $g(x_0) \neq y_0$ for all $x_0 \in \{0,1\}^m$, it suffices to find a $y \in \{0,1\}^{m+t}$ such that $h_e(x) \neq y$ for all $x \in \{0,1\}^m$. Consider the following stronger claim:

Claim A.3 (In S^1_2)**.** *For all $m, i \in \mathrm{Log}$, all $y \in \{0,1\}^m$, all $v \in \{0,1\}^i$, and all $z \in \{0,1\}^m$, if $h_{e,i}(z) = y{:}v$, then $h_{e,i+1}(z) = g(y){:}v$. In particular, for all $y_0 \in \{0,1\}^{m+1}$ and all $v \in \{0,1\}^i$, if the concatenated string $y_0{:}v$ is outside the range of $h_{e,i+1}$, then either y_0 is outside the range of g_d, or any $w \in \{0,1\}^m$ that witnesses the opposite by satisfying $g_d(w) = y_0$ is such that the concatenated string $w{:}v$ is outside the range of $h_{e,i}$.*

Proof. The first statement is a direct consequence of the recursive definition of the $h_{e,i}$. To prove the second statement, assume that there exists $w \in \{0,1\}^m$ such that $g_d(w) = y_0$ yet $w{:}v$ is inside the range of $h_{e,i}$, say $h_{e,i}(u) = w{:}v$ for some $u \in \{0,1\}^m$. Then, by the first part of the lemma we get $h_{e,i+1}(u) = g_d(w){:}v = y_0{:}v$, which means that $y_0{:}v$ is inside the range of $h_{e,i+1}$. □

In case $i = 0$, the second part of Lemma A.3 concludes that y_0 is outside the range of $h_{e,1}$, which equals g_d. To see this observe that if $i = 0$, then v is the empty string, and any $w \in \{0,1\}^m$ is always *inside* the range of $h_{e,0}$, since $h_{e,0}$ is the identity map on $\{0,1\}^m$. This means that, when given a $y \in \{0,1\}^{m+i}$ that is outside the range of $h_{e,i}$ with $i > 0$, the following $\mathsf{FP}^{\mathsf{NP}}[\mathsf{wit}]$ procedure (for now making more than 2 queries) halts in less than i iterations and finds a $y_0 \in \{0,1\}^{m+1}$ that is outside the range of g_d:

1. given $y = (a_1, \ldots, a_{m+i}) \in \{0,1\}^{m+i}$,
2. set $y_0 := (a_1, \ldots, a_{m+1})$,
3. for $j = 0, 1, \ldots, i-2$ do the following:
4. query "$\exists w{\in}\{0,1\}^m\ g_d(w){=}y_j$" to the NP-oracle
5. if answer is NO, halt and output y_j,
6. if answer is YES, get such w and set $y_{j+1} := w{:}a_{m+j+2}$,
7. halt and fail.

In turn, this $\mathsf{FP}^{\mathsf{NP}}[\mathsf{wit}]$-procedure is special in that it halts after it gets the first NO. This means that it can be replaced by the following different but equivalent $\mathsf{FP}^{\mathsf{NP}}[\mathsf{wit}, 2]$-procedure:

1. given $y = (a_1, \ldots, a_{m+i}) \in \{0,1\}^{m+i}$,
2. set $y_0 := (a_1, \ldots, a_{m+1})$,
3. query "$\exists k{<}i{-}1\ \exists y_1, \ldots, y_k\ \exists w_0, \ldots, w_{k-1}\ \forall j{<}k\ (g_d(w_j) = y_j \wedge y_{j+1} = w_j{:}a_{m+j+2})$"
6. if answer is NO, halt and fail,
7. if answer is YES, get such $k, y_1, \ldots, y_k, w_0, \ldots, w_{k-1}$,
8. query "$\exists w_k\ g_d(w_k){=}y_k$"
9. if answer is NO, halt and output y_k,
10. halt and fail.

We now show that the desired claim follows. Let $s_{e,i}$ denote the $\mathsf{FP}^{\mathsf{NP}}[\mathsf{wit}, 2]$-machine defined above; precisely, s is a $\mathsf{PV}_2[\mathsf{wit}, 2]$-symbol in the theory, and $s_{e,i}$ is its parameterization with e, i.

Claim A.4 (In S^1_2)**.** *For all $m, i \in \mathrm{Log}$ such that $i > 0$ and all $y \in \{0,1\}^{m+i}$, if $y \in \{0,1\}^{m+i}$ is outside the range of $h_{e,i}$, then there exists a computation of $s_{e,i}(y)$ that does not fail, and any such computation outputs a string in $\{0,1\}^{m+1}$ that is outside the range of g.*

Proof. For fixed positive $m, i \in \mathrm{Log}$ and fixed $y = (a_1, \ldots, a_{m+i}) \in \{0,1\}^{m+i}$ the strategy for this proof is a maximization argument showing that a longest computation of $s_{e,i}(y)$ exists. We exploit the maximality to argue that this computation does not fail and also that any such computation outputs a correct solution. The maximization argument is proved by using the *length-maximization* principle Σ^b_1-$\mathsf{LENGTH\text{-}MAX}$, which is available in S^1_2 (see Lemma 5.2.7 in [26]).

Consider the statement $\phi(x)$ asserting of $k = |x|$ that there exist $y_0, y_1, \ldots, y_k \in \{0,1\}^{m+1}$ and $w_0, w_1, \ldots, w_{k-1} \in \{0,1\}^m$ that witness the first oracle query in the algorithm for $s_{e,i}$. Formally:

$$\begin{aligned} \phi(x) \;:=\; & \exists k{\leq}x\ (k{=}|x| \wedge \exists y_0, y_1, \ldots, y_k{\in}\{0,1\}^{m+1}\ \exists w_0, w_1, \ldots, w_{k-1}{\in}\{0,1\}^m \\ & y_0{=}y[1, m+1] \wedge \forall j{<}k\ (g_d(w_j){=}y_j \wedge y_{j+1}{=}w_j{:}a_{m+j+2})). \end{aligned} \tag{48}$$

This is a Σ^b_1-formula with $m, i, y = (a_1, \ldots, a_{m+i})$ as parameters: the sequences in it have the length of a length, and the $\forall j{<}k$ quantifier in it is sharply bounded (by $k = |x|$). We intend to apply the length-maximization principle ϕ-$\mathsf{LENGTH\text{-}MAX}$ with bound $2^{i-1} - 1$. Recall that $i > 0$. The existence of a length-maximum $x \leq 2^{i-1} - 1$ such that $\phi(x)$ holds will let us argue that there is a computation of $s_{e,i}(y)$ that does not fail, and also that any such computation outputs a string in $\{0,1\}^{m+1}$ that is outside the range of g_d.

First, note that $\phi(0)$ holds by setting $k = 0$ (recall that $|0| = 0$) and $y_0 = y[1, m+1]$, and by setting $w_0, w_1, \ldots, w_{k-1}$ to the empty sequence of strings; the quantifier $\forall i{<}k$ holds vacuously in this case. By ϕ-$\mathsf{LENGTH\text{-}MAX}$ applied to the upper bound $2^{i-1}-1$, there exists a length maximum $x \leq 2^{i-1} - 1$ such that $\phi(x)$ holds. Let $k = |x|$ and note that $k \leq i - 1$ since every number below 2^{i-1} has length $i-1$ or less. By the definition of the procedure $s_{e,i}$ and the maximality of $x \leq 2^{i-1} - 1$, which could have length up to $i - 1$, if $k < i - 1$ and $y_0, y_1, \ldots, y_k$ are the witnesses for $\phi(x)$, then $s_i(\overline{y})$ does not fail and outputs y_k. In addition, in such a case the NP-oracle answered NO on the query "$\exists w_k{\in}\{0,1\}^m\ g(w_k){=}y_k$", and hence $s_{e,i}(y) = y_k$ is the string outside the range of g_d we were looking for. To complete the proof it remains to be seen that, indeed, $k < i - 1$.

Suppose the contrary, so $k = i - 1$ and hence $|x| = i - 1$. Let $y_0, y_1, \ldots, y_{i-1}$ and $w_0, w_1, \ldots, w_{i-2}$ be the witnesses for $\phi(x)$. For $j = 0, 1, 2, \ldots, i{-}1$, let $v_j = (a_{m+j+2}, \ldots, a_{m+i})$; note that v_j has length $i - j - 1$ and that v_{i-1} is the empty string. Consider the sequence of concatenations $y_j{:}v_j$ for $j = 0, 1, 2, \ldots, i - 1$. Note that $y_0{:}v_0$ is y. By assumption y is outside the range of $h_{e,i}$. Since y_0 is in the range of g_d and w_0 witnesses it, by the second part of Lemma A.3 we conclude that $w_0{:}v_0$ is outside the range of $h_{e,i-1}$. Now note that

$w_0{:}v_0$ equals $y_1{:}v_1$, so we have shown that $y_1{:}v_1$ is outside the range of $h_{e,i-1}$. More generally, consider the Π^b_1-formula

$$\psi(x) := \forall j{\leq}x(j{=}|x| \wedge j{\leq}i{-}1 \to \forall w{\in}\{0,1\}^m\ h_{e,i-j}(w) \neq y_j{:}v_j), \tag{49}$$

with i among others as parameter. This says of the length $j := |x|$ that $y_j{:}v_j$ is outside the range of $h_{e,i-j}$. We know that $\psi(0)$ holds by assumption since $y = y_0{:}v_0$ and y is outside the range of $h_{e,i}$ (and recall $|0| = 0$). Further, the second statement in Lemma A.3 combined with the fact that $g_d(w_j) = y_j$ holds for all $j < i-1$ shows that $\psi(\lfloor x/2 \rfloor)$ implies $\psi(x)$, for all x. By Π^b_1-PIND we get that $\psi(2^{i-1}-1)$ holds. But v_{i-1} is the empty string, so $\psi(2^{i-1}-1)$ says that y_{i-1} is outside the range of $h_{e,1}$, which is absurd since $y_{i-1} = g_d(w_{i-2})$ and $h_{e,1}$ is g_d. □

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