499 Modal and Temporal Logic *Autumn 2008*

Tutorial Exercises 1 (mjs)

1. Let Σ be a (not necessarily normal) modal logic closed under the rule RM:

   RM. $$\frac{A \rightarrow B}{\Box A \rightarrow \Box B}$$

   Prove that C is a theorem of Σ if and only if K is. The schemas C and K are:

   C. $(\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$

   K. $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

2. Prove that the following are theorems of any modal logic closed under RM (and hence also of any normal logic):

   (a) $\Box A \rightarrow \Box(B \rightarrow A)$
   (b) $\Box\neg A \rightarrow \Box(A \rightarrow B)$
   (c) $(\Box A \vee \Box B) \rightarrow \Box(A \vee B)$
   (d) $\Diamond(A \wedge B) \rightarrow (\Diamond A \wedge \Diamond B)$
   (e) $\Diamond(A \rightarrow B) \vee \Box(B \rightarrow A)$
   (f) $(\Box A \rightarrow \Diamond B) \rightarrow \Diamond(A \rightarrow B)$
   (g) $(\Box A \rightarrow \Diamond A) \rightarrow \Diamond\top$
   (h) $(\Diamond A \rightarrow \Box B) \rightarrow \Box(A \rightarrow B)$

   You may find it helpful to note that the following are all tautologies:

   - $A \rightarrow (B \rightarrow A)$
   - $\neg A \rightarrow (A \rightarrow B)$
   - $A \rightarrow (A \vee B)$
   - $(A \rightarrow B) \vee (B \rightarrow A)$

   You should check the above if they are not obvious. (The last one is perhaps a bit surprising.)

3. Prove the following statements given in the notes (under 'Example').

   (i) The set of all formulas is a system of modal logic, the *inconsistent logic*.
   (ii) If $\{\Sigma_i \mid i \in I\}$ is a collection of logics, then $\bigcap_{i \in I} \Sigma_i$ is a logic.
   (iii) Define Σ_{F} to be the set of formulas valid in a class F of frames. Σ_{F} is a logic.

   Parts (i) and (ii) are in the lecture notes. For part (iii), closure under US is sketched in the lecture notes. So it just remains to show that Σ_{F} contains *PL* and is closed under MP.

4. Prove the following statements (also given in the notes under 'Example')

   (i) The inconsistent logic is a normal logic.
   (ii) *PL* is not a normal logic.
   (iii) If $\{\Sigma_i \mid i \in I\}$ is a collection of normal logics, then $\bigcap_{i \in I} \Sigma_i$ is a normal logic.
   (iv) If F is any class of relational ('Kripke') frames then Σ_{F}, the set of formulas valid in F, is a normal logic.

5. Prove that every normal logic has the following rules of inference and theorems:

   RM$\Diamond$. $$\frac{A \rightarrow B}{\Diamond A \rightarrow \Diamond B}$$

   N$\Diamond$. $\neg\Diamond\bot$

   MC$\Diamond$. $\Diamond(A \vee B) \leftrightarrow (\Diamond A \vee \Diamond B)$

6. Prove that the normal logic *KT5* is the same as the normal logic *KT45*. (Both are the logic called S5.)

   What you need to do is to show that *KT5* contains all instances of the schema 4:

   4. $\Box A \rightarrow \Box\Box A$

   Schemas T and 5 are as follows:

   T. $\Box A \rightarrow A$

   5. $\Diamond A \rightarrow \Box\Diamond A$

   Hint: From T and 5 you can derive $A \rightarrow \Box\Diamond A$.
   From 5 you can derive $\Box\Diamond\Box A \rightarrow \Box\Box A$.

7. *(from 2003 exam)*

   Consider a language with modal operators O and $\Box$. Models are Kripke structures $\langle W, R_{\mathrm{O}}, R_{\Box}, h \rangle$ where W is a set of worlds, h is a valuation function for the atoms, and R_{O} and $R_{\Box}$ are the accessibility relations for the operators O and $\Box$ respectively.

   Suppose further that O is a normal modality of type KD (R_{O} is serial) and $\Box$ is a normal modality of type KT ($R_{\Box}$ is reflexive).

   Suppose now that the language is extended with another modal operator Oblig defined as follows:
   $$\text{Oblig}\, A \stackrel{\text{def}}{=} \mathrm{O}A \wedge \neg\Box A$$

   Show that Oblig has the following properties

   noN. $\neg\,\text{Oblig}\,\top$
   D. $\text{Oblig}\, A \rightarrow \neg\,\text{Oblig}\,\neg A$, i.e. $\neg(\text{Oblig}\, A \wedge \text{Oblig}\,\neg A)$
   C. $(\text{Oblig}\, A \wedge \text{Oblig}\, B) \rightarrow \text{Oblig}(A \wedge B)$

   You may use any standard properties of normal logics, such as $\Box(A \wedge B) \rightarrow \Box A$, as long as you identify them clearly.

/MORE OVERLEAF ...

8. Prove that the following are theorems of every classical *ET5* system:

P.	$\Diamond\top$
N.	$\Box\top$

The next two questions are more demanding. You might want to consult the solutions as you go along.

9. Show that the following 'reduction laws' are all theorems of the normal system S4 (=*KT4*):

$$\Box A \leftrightarrow \Box\Box A \qquad \Diamond A \leftrightarrow \Diamond\Diamond A$$

$$\Diamond\Box A \leftrightarrow \Diamond\Box\Diamond\Box A \qquad \Box\Diamond A \leftrightarrow \Box\Diamond\Box\Diamond A$$

Now show that every modality (i.e., every sequence of $\Box$, $\Diamond$ and $\neg$, in any order) is equivalent in S4 to one of 14 distinct modalities.

10. By comparison with the previous question, try to identify the reduction laws for the system S5 (=*KT45* = *KT5*), and hence determine how many distinct modalities there are in S5.