499 Modal and Temporal Logic

Deducibility, Consistency, Maxi-consistent sets

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Further reading:
B.F. Chellas, *Modal logic: an introduction*. Cambridge University Press, 1980.
P. Blackburn, M. de Rijke, Y. Venema, Chapter 4, *Modal Logic*. Cambridge University Press, 2002.

Reminder

– The set of formulas Σ is a *system of modal logic* iff it contains all propositional tautologies (*PL*) and is closed under modus ponens (MP) and uniform substitution (US).
– $\vdash_\Sigma A$ means that A is a theorem of Σ. $\vdash_\Sigma A$ iff $A \in \Sigma$.

(The following is applicable to all systems of normal logic, not just normal systems.)

Deducibility and consistency

A formula A is *deducible* from a set of formulas Γ in a logic Σ — written $\Gamma \vdash_\Sigma A$ — iff Σ contains a theorem of the form

$$(A_1 \wedge \cdots \wedge A_n) \rightarrow A$$

where the conjuncts $A_1, \ldots, A_n$ are formulas in Γ. It is convenient to extend the notation: for Γ' a set of formulas, $\Gamma \vdash_\Sigma \Gamma'$ means that $\Gamma \vdash_\Sigma A$ for every A in Γ'.
A set of formulas Γ is inconsistent in Σ (Σ-inconsistent) just in case $\perp$ is Σ-deducible from Γ. A set of formulas Σ-consistent when it is not Σ-inconsistent.

Definition 1 (Deducibility) $\Gamma \vdash_\Sigma A$ *iff there are formulas* $A_1, \ldots, A_n \in \Gamma$ $(n \geq 0)$ *such that* $\vdash_\Sigma (A_1 \wedge \cdots \wedge A_n) \rightarrow A$.
For Γ' *a set of formulas,* $\Gamma \vdash_\Sigma \Gamma'$ *means that* $\Gamma \vdash_\Sigma A$ *for every* A *in* Γ'.

Definition 2 (Consistency) Γ *is* Σ*-consistent iff not* $\Gamma \vdash_\Sigma \perp$. Γ *is* Σ*-inconsistent iff* $\Gamma \vdash_\Sigma \perp$.

Some properties (there is no need to memorize these theorems!):

Theorem 3 *[Chellas Thm 2.16, p47]*

(1) $\vdash_\Sigma A$ *iff* $\emptyset \vdash_\Sigma A$.
(2) $\vdash_\Sigma A$ *iff for every* Γ, $\Gamma \vdash_\Sigma A$.
(3) *If* $\Gamma \vdash_{PL} A$, *then* $\Gamma \vdash_\Sigma A$.
(4) *If* $A \in \Gamma$ *then* $\Gamma \vdash_\Sigma A$. *(Or using the* $\vdash_\Sigma$ *notation for sets of formulas,* $\Gamma \vdash_\Sigma \Gamma$.*)*
(5) *If* $\Gamma \vdash_\Sigma B$ *and* $\{B\} \vdash_\Sigma A$, *then* $\Gamma \vdash_\Sigma A$.
More generally, for Γ' *any set of formulas: if* $\Gamma \vdash_\Sigma \Gamma'$ *and* $\Gamma' \vdash_\Sigma A$, *then* $\Gamma \vdash_\Sigma A$.
(6) *If* $\Gamma \vdash_\Sigma A$ *and* $\Gamma \subseteq \Gamma'$, *then* $\Gamma' \vdash_\Sigma A$.
(7) $\Gamma \vdash_\Sigma A$ *iff there is a finite subset* Γ_x *of* Γ *such that* $\Gamma_x \vdash_\Sigma A$.
(8) $\Gamma \vdash_\Sigma A \rightarrow B$ *iff* $\Gamma \cup \{A\} \vdash_\Sigma B$.

Comments on Theorem 3

Properties (1)—(3) should be clear enough.
Property (4) is reflexivity of the deducibility relation $\vdash_\Sigma$. It's sometimes called 'inclusion'.
Property (5) is transitivity of the deducibility relation $\vdash_\Sigma$. It's sometimes called 'cut'.
Property (6) means that the deducibility relation $\vdash_\Sigma$ is *monotonic*. It can be expressed as $\Gamma \vdash_\Sigma A \;\Rightarrow\; \Gamma \cup \Gamma' \vdash_\Sigma A$, for any set of formulas Γ'.
Property (7) is 'compactness' of the deducibility relation $\vdash_\Sigma$.
Property (8) is the so-called *deduction theorem* for $\vdash_\Sigma$.

Proofs:

(1) $\vdash_\Sigma A$ iff $\emptyset \vdash_\Sigma A$.
Trivially: if $\vdash_\Sigma A$ then there is a Σ-theorem of the form $(A_1 \wedge \cdots \wedge A_n) \rightarrow A$ where $n = 0$ and the conditional is just A. Since the (non-existent) A_i in the antecedent are all in $\emptyset$, $\emptyset \vdash_\Sigma A$. Conversely, if $\emptyset \vdash_\Sigma A$ then it must be that $\vdash_\Sigma (A_1 \wedge \cdots \wedge A_n) \rightarrow A$ for $n = 0$. That is, $\vdash_\Sigma A$.
(2) $\vdash_\Sigma A$ iff for every Γ, $\Gamma \vdash_\Sigma A$.
Left-to-right: as for part (1), if $\vdash_\Sigma A$ then there is a Σ-theorem of the form $(A_1 \wedge \cdots \wedge A_n) \rightarrow A$ where $n = 0$. Since the (non-existent) A_i in the antecedent are trivially all in Γ, for any set of formulas Γ, $\Gamma \vdash_\Sigma A$. For the converse, if $\Gamma \vdash_\Sigma A$ for any set of formulas Γ, then in particular $\emptyset \vdash_\Sigma A$, which by part (1) means $\vdash_\Sigma A$.
(3) If $\Gamma \vdash_{PL} A$, then $\Gamma \vdash_\Sigma A$.
If $\Gamma \vdash_{PL} A$ then there is a theorem $(A_1 \wedge \cdots \wedge A_n) \rightarrow A$ in *PL* where $\{A_1, \ldots, A_n\} \subseteq \Gamma$. But $PL \subseteq \Sigma$ for any system Σ, so also $\Gamma \vdash_\Sigma A$.
(4) If $A \in \Gamma$ then $\Gamma \vdash_\Sigma A$. (Or $\Gamma \vdash_\Sigma \Gamma$.)
The formula $A \rightarrow A$ is a tautology, hence a *PL*-theorem, hence a Σ-theorem for any system Σ. So if $A \in \Gamma$ then there is a theorem $A \rightarrow A$ in Σ whose antecedent A is in Γ. So $\Gamma \vdash_\Sigma A$.

(5) If $\Gamma \vdash_\Sigma B$ and $\{B\} \vdash_\Sigma A$, then $\Gamma \vdash_\Sigma A$.

More generally: if $\Gamma \vdash_\Sigma \Gamma'$ and $\Gamma' \vdash_\Sigma A$, then $\Gamma \vdash_\Sigma A$.

The first part is obviously a special case of the more general statement. So suppose $\Gamma \vdash_\Sigma \Gamma'$ and $\Gamma' \vdash_\Sigma A$. $\Gamma' \vdash_\Sigma A$ means there is a theorem $(A_1 \wedge \cdots \wedge A_n) \rightarrow A$ in Σ such that $\{A_1, \ldots, A_n\} \subseteq \Gamma'$. $\Gamma \vdash_\Sigma \Gamma'$ means $\Gamma \vdash_\Sigma B$ for every $B \in \Gamma'$, and so in particular $\Gamma \vdash_\Sigma A_i$ for every A_i $(1 \leq i \leq n)$. $\Gamma \vdash_\Sigma A_i$ for each such A_i means there is a Σ-theorem $(A_1^i \wedge \cdots \wedge A_{m_i}^i) \rightarrow A_i$ for each A_i such that $\{A_1^i, \ldots, A_{m_i}^i\} \subseteq \Gamma$. By RPL, there is therefore a Σ-theorem $(A_1^1 \wedge \cdots \wedge A_{m_1}^1 \wedge \cdots \wedge A_1^i \wedge \cdots \wedge A_{m_i}^i \wedge \cdots \wedge A_1^n \wedge \cdots \wedge A_{m_n}^n) \rightarrow (A_1 \wedge \cdots \wedge A_n)$, and hence also a Σ-theorem $(A_1^1 \wedge \cdots \wedge A_{m_1}^1 \wedge \cdots \wedge A_1^i \wedge \cdots \wedge A_{m_i}^i \wedge \cdots \wedge A_1^n \wedge \cdots \wedge A_{m_n}^n) \rightarrow A$.Since $\{A_1^1, \ldots, A_{m_1}^1, \ldots, A_1^i, \ldots, A_{m_i}^i, \ldots A_1^n, \ldots, A_{m_n}^n\} \subseteq \Gamma$, we have $\Gamma \vdash_\Sigma A$.

(6) If $\Gamma \vdash_\Sigma A$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash_\Sigma A$.

Monotonicity. Very easy: if $\Gamma \vdash_\Sigma A$ then there is a Σ-theorem of the form $(A_1 \wedge \cdots \wedge A_n) \rightarrow A$ such that $\{A_1, \ldots, A_n\} \subseteq \Gamma$. But if $\Gamma \subseteq \Gamma'$ then also $\{A_1, \ldots, A_n\} \subseteq \Gamma'$, and $\Gamma' \vdash_\Sigma A$ as required.

(7) $\Gamma \vdash_\Sigma A$ iff there is a finite subset Γ_x of Γ such that $\Gamma_x \vdash_\Sigma A$.

Compactness. Left-to-right follows immediately from the fact that by definition the number of conjuncts in the antecedent of the required conditional $(A_1 \wedge \cdots \wedge A_n) \rightarrow A$ is finite. Right-to-left follows from part (6) (monotonicity).

(8) $\Gamma \vdash_\Sigma A \rightarrow B$ iff $\Gamma \cup \{A\} \vdash_\Sigma B$.

Deduction theorem:

$$\begin{aligned}
\Gamma \vdash_\Sigma A \rightarrow B \quad &\Leftrightarrow \quad \vdash_\Sigma (A_1 \wedge \cdots \wedge A_n) \rightarrow (A \rightarrow B) \quad \text{for some } \{A_1, \ldots, A_n\} \subseteq \Gamma \\
&\Leftrightarrow \quad \vdash_\Sigma (A_1 \wedge \cdots \wedge A_n \wedge A) \rightarrow B \quad \text{for some } \{A_1, \ldots, A_n\} \subseteq \Gamma, \text{ by RPL} \\
&\Leftrightarrow \quad \vdash_\Sigma (A_1 \wedge \cdots \wedge A_n \wedge A) \rightarrow B \quad \text{for some } \{A_1, \ldots, A_n, A\} \subseteq \Gamma \cup \{A\} \\
&\Leftrightarrow \quad \Gamma \cup \{A\} \vdash_\Sigma B
\end{aligned}$$

Theorem 4 *[Chellas Thm 2.16, p47]*

(1) Γ *is* Σ*-consistent iff there is an* A *such that not* $\Gamma \vdash_\Sigma A$.
(2) Γ *is* Σ*-consistent iff there is no* A *such that both* $\Gamma \vdash_\Sigma A$ *and* $\Gamma \vdash_\Sigma \neg A$.
(3) *If* Γ *is* Σ*-consistent, then* Γ *is PL-consistent.*
(4) *If* Γ *is* Σ*-consistent and* $\Gamma' \subseteq \Gamma$*, then* Γ' *is* Σ*-consistent.*
(5) Γ *is* Σ*-consistent iff every finite subset* Γ_x *of* Γ *is* Σ*-consistent.*
(6) $\Gamma \vdash_\Sigma A$ *iff* $\Gamma \cup \{\neg A\}$ *is* Σ*-inconsistent.*
(7) $\Gamma \cup \{A\}$ *is* Σ*-consistent iff* $\Gamma \nvdash_\Sigma \neg A$.

Comments on Theorem 4

Properties (1)–(2) are alternative (equivalent) characterisations of Σ-consistency of a set of formulas Γ.

Properties (3)–(5) should be clear enough given the corresponding properties of $\vdash_\Sigma$.

Properties (6)–(7) relate Σ-consistency and deducibility $\vdash_\Sigma$.

Proofs: All of these follow more or less immediately from their counterparts in Theorem 3.

(1) Γ is Σ-consistent iff there is an A such that not $\Gamma \vdash_\Sigma A$.

Suppose that Γ is Σ-consistent, i.e. that not $\Gamma \vdash_\Sigma \bot$. Then clearly there is a formula A such that not $\Gamma \vdash_\Sigma A$. For the reverse, suppose that Γ is Σ-inconsistent, i.e. that $\Gamma \vdash_\Sigma \bot$. Then by RPL and Theorem 3(3), $\{\bot\} \vdash_\Sigma A$, for every formula A. So $\Gamma \vdash_\Sigma A$ for every formula A by Theorem 3(5).

(2) Γ is Σ-consistent iff there is no A such that both $\Gamma \vdash_\Sigma A$ and $\Gamma \vdash_\Sigma \neg A$.

Prove the contrapositive: that Γ is Σ-inconsistent iff there is a formula A such that $\Gamma \vdash_\Sigma A$ and $\Gamma \vdash_\Sigma \neg A$. Left-to-right of this follows from part (1). For right-to-left: $\Gamma \vdash_\Sigma \{A, \neg A\}$ and $\{A, \neg A\} \vdash_{PL} \bot$ implies $\Gamma \vdash_\Sigma \bot$ by Theorem 3 parts (3) and (5).

(3) If Γ is Σ-consistent, then Γ is *PL*-consistent.

Prove the contrapositive: if Γ is *PL*-inconsistent then $\Gamma \vdash_{PL} \bot$, which implies by Theorem 3(3) that $\Gamma \vdash_\Sigma \bot$, i.e. that Γ is Σ-inconsistent.

(4) If Γ is Σ-consistent and $\Gamma' \subseteq \Gamma$, then Γ' is Σ-consistent.

Again, prove the contrapositive: if $\Gamma \subseteq \Gamma'$ then $\Gamma \vdash_\Sigma \bot$ implies $\Gamma' \vdash_\Sigma \bot$ by Theorem 3(6) (monotonicity of $\vdash_\Sigma$).

(5) Γ is Σ-consistent iff every finite subset Γ_x of Γ is Σ-consistent.

Follows straightforwardly from Theorem 3(7).

(6) $\Gamma \vdash_\Sigma A$ iff $\Gamma \cup \{\neg A\}$ is Σ-inconsistent.

Left-to-right: suppose $\Gamma \vdash_\Sigma A$. By Theorem 3(6) (monotonicity of $\vdash_\Sigma$), we have $\Gamma \cup \{\neg A\} \vdash_\Sigma A$. But by Theorem 3(4) (reflexivity of $\vdash_\Sigma$), we have $\Gamma \cup \{\neg A\} \vdash_\Sigma \neg A$. So by part (1) Γ is Σ-inconsistent.

Right-to-left: suppose $\Gamma \cup \{\neg A\}$ is Σ-inconsistent, i.e. that $\Gamma \cup \{\neg A\} \vdash_\Sigma \bot$. Then by Theorem 3(8) (deduction theorem for $\vdash_\Sigma$), $\Gamma \vdash_\Sigma \neg A \rightarrow \bot$. But $\neg A \rightarrow \bot$ is equivalent in *PL* to A, so by Theorem 3 parts (3) and (5), $\Gamma \vdash_\Sigma A$.

(7) $\Gamma \cup \{A\}$ is Σ-consistent iff $\Gamma \nvdash_\Sigma \neg A$.

Follows straightforwardly from part (6).

Maxi-consistent sets

A set of sentences is *maximal consistent* in a system Σ (Σ-maxi-consistent for short) just in case it is Σ-consistent and has only Σ-inconsistent proper extensions. In other words, a set is Σ-maxi-consistent if it is consistent and contains as many formulas as it can without becoming inconsistent.

Definition 5 (Σ-maxi-consistent set) *A set of formulas Γ is* Σ-maxi-consistent *iff (i) Γ is Σ-consistent, and (ii) for every formula A, if $\Gamma \cup \{A\}$ is Σ-consistent, then $A \in \Gamma$.*

Note that clause (ii) says that where Γ is Σ-maxi-consistent, the addition of a formula not already in Γ yields a Σ-inconsistent set of formulas.

Here are some properties of Σ-maxi-consistent sets.

Theorem 6 *[Chellas Thm 2.18, p53] Let Γ be a Σ-maxi-consistent set. Then:*

(1) $A \in \Gamma \Leftrightarrow \Gamma \vdash_\Sigma A$.
(2) $\Sigma \subseteq \Gamma$.
(3) $\top \in \Gamma$.
(4) $\bot \notin \Gamma$.
(5) $\neg A \in \Gamma \Leftrightarrow A \notin \Gamma$.
(6) $A \wedge B \in \Gamma \Leftrightarrow A \in \Gamma$ *and* $B \in \Gamma$.
(7) $A \vee B \in \Gamma \Leftrightarrow A \in \Gamma$ *or* $B \in \Gamma$.
(8) $A \to B \in \Gamma \Leftrightarrow (A \in \Gamma \Rightarrow B \in \Gamma)$.
(9) $A \leftrightarrow B \in \Gamma \Leftrightarrow (A \in \Gamma \Leftrightarrow B \in \Gamma)$.
(10) Γ *is a Σ-system.*

Proof I hesitate to show all the proofs because the details, in particular of (6)–(9) are rather fiddly, and can obscure what is essentially a simple argument. Still ...

(1) $A \in \Gamma \Leftrightarrow \Gamma \vdash_\Sigma A$.

Left-to-right is just Theorem 3(4). For right-to-left: suppose not, i.e., suppose that $\Gamma \vdash_\Sigma A$ but $A \notin \Gamma$. By the maximality of Γ, $\Gamma \cup \{A\}$ is Σ-inconsistent. From this by Theorem 4(6), $\Gamma \vdash_\Sigma \neg A$. So Γ is Σ-inconsistent (Theorem 4(2)). But this contradicts Γ is Σ-maxi-consistent.

(2) $\Sigma \subseteq \Gamma$.

Suppose that $A \in \Sigma$, i.e. that $\vdash_\Sigma A$. Then by Theorem 3, $\Gamma' \vdash_\Sigma A$ for every set of formulas Γ'. In particular, $\Gamma \vdash_\Sigma A$ which by part (1) above means $A \in \Gamma$.

(3) $\top \in \Gamma$.

$\top \in PL$, so $\top \in \Sigma$, so $\top \in \Gamma$ by the previous part (2).

(4) $\bot \notin \Gamma$.

Suppose $\bot \in \Gamma$. Then $\Gamma \vdash_\Sigma \bot$, which contradicts Γ is Σ-maxi-consistent.

(5) $\neg A \in \Gamma \Leftrightarrow A \notin \Gamma$.

Suppose not, i.e., suppose that either (i) $A \in \Gamma$ and $\neg A \in \Gamma$ or (ii) $A \notin \Gamma$ and $\neg A \notin \Gamma$. If (i), then by Theorem 4(2), Γ is Σ-inconsistent, which is a contradiction. If (ii), then by part (1), $\Gamma \nvdash_\Sigma A$ and $\Gamma \nvdash_\Sigma \neg A$ which means (by Theorem 4(7)) $\Gamma \cup \{A\}$ is Σ-consistent and $\Gamma \cup \{\neg A\}$ is Σ-consistent. So by maximality of Γ, $A \in \Gamma$ and $\neg A \in \Gamma$. But that again contradicts that Γ is Σ-consistent.

(6) $A \wedge B \in \Gamma \Leftrightarrow A \in \Gamma$ and $B \in \Gamma$.

For left-to-right: suppose $A \wedge B \in \Gamma$. Then by part (1) $\Gamma \vdash_\Sigma A \wedge B$. Now $\{A \wedge B\} \vdash_{PL} A$ and hence $\{A \wedge B\} \vdash_\Sigma A$, so by Theorem 3(5) (transitivity of $\vdash_\Sigma$) we have $\Gamma \vdash_\Sigma A$, from which $A \in \Gamma$ by part (1). The argument for $B \in \Gamma$ is similar.

For right-to-left, by a similar argument: $A \in \Gamma$ and $B \in \Gamma$ imply $\Gamma \vdash_\Sigma A$ and $\Gamma \vdash_\Sigma B$, i.e., $\Gamma \vdash_\Sigma \{A, B\}$. $\{A, B\} \vdash_{PL} A \wedge B$ and so $\{A, B\} \vdash_\Sigma A \wedge B$. By the general form of Theorem 3(5) (transitivity of $\vdash_\Sigma$), $\Gamma \vdash_\Sigma A \wedge B$, from which $A \wedge B \in \Gamma$ by part (1).

(7) $A \vee B \in \Gamma \Leftrightarrow A \in \Gamma$ or $B \in \Gamma$.

Right-to-left: $A \in \Gamma$ implies $\Gamma \vdash_\Sigma$, and $\{A\} \vdash_{PL} A \vee B$. The rest follows as in part (6) above.

For left-to-right, we show that $A \vee B \in \Gamma$ and $A \notin \Gamma$ implies $B \in \Gamma$. Since Γ is Σ-maxi-consistent, $A \notin \Gamma$ implies $\neg A \in \Gamma$ by part (5). And by part (1), we have $\Gamma \vdash_\Sigma \{A \vee B, \neg A\}$. Now $\{A \vee B, \neg A\} \vdash_{PL} B$, so $\Gamma \vdash_\Sigma B$, and hence $B \in \Gamma$ by part (1).

(8) $A \to B \in \Gamma \Leftrightarrow (A \in \Gamma \Rightarrow B \in \Gamma)$.

Left-to-right follows by a similar argument to parts (6) and (7). We need to show that if $A \to B \in \Gamma$ and $A \in \Gamma$ then $B \in \Gamma$, i.e. by part (1) that $\Gamma \vdash_\Sigma \{A \to B, A\}$ implies $\Gamma \vdash_\Sigma B$. This follows as in parts (6) and (7) because $\{A \to B, A\} \vdash_{PL} B$.

For right-to-left we show that $A \to B \notin \Gamma$ implies $A \in \Gamma$ and $B \notin \Gamma$. By parts (5) and (1) it is enough to show $\Gamma \vdash_\Sigma \neg(A \to B)$ implies $\Gamma \vdash_\Sigma A$ and $\Gamma \vdash_\Sigma \neg B$. And this follows as in previous parts from $\{\neg(A \to B)\} \vdash_{PL} A$ and $\{\neg(A \to B)\} \vdash_{PL} \neg B$. (Note: $\neg(A \to B)$ is equivalent in *PL* to $A \wedge \neg B$).

(9) $A \leftrightarrow B \in \Gamma \Leftrightarrow (A \in \Gamma \Leftrightarrow B \in \Gamma)$.

This obviously follows from part (8), since $A \leftrightarrow B$ is equivalent in *PL* as $(A \to B) \wedge (B \to A)$.

(10) Γ is a Σ-system.

This is just a re-statement of part (2). Γ is a Σ-system means that Γ contains every theorem of Σ, or in other words, $\Sigma \subseteq \Gamma$.

Lindenbaum's Lemma

Theorem 7 (Lindenbaum's lemma) *Let Γ be a Σ-consistent set of formulas. Then there exists a Σ-maxi-consistent set Δ such that $\Gamma \subseteq \Delta$.*

Proof *(Sketch)* Let $A_0, A_1, A_2, \ldots$ be an enumeration of the formulas of the language. Define the set Δ as the union of a sequence of Σ-consistent sets, as follows:

$$\begin{aligned} \Delta_0 &= \Gamma, \\ \Delta_{i+1} &= \begin{cases} \Delta_i \cup \{A_i\}, & \text{if this is } \Sigma\text{-consistent} \\ \Delta_i \cup \{\neg A_i\}, & \text{otherwise} \end{cases} \\ \Delta &= \bigcup_{i \geq 0} \Delta_i. \end{aligned}$$

Now it remains to show that

(i) Δ_i is Σ-consistent, for all i;

(ii) exactly one of A and $\neg A$ is in Δ, for every formula A;

(iii) if $\Delta \vdash_\Sigma A$, then $A \in \Delta$; and finally

(iv) Δ is a Σ-maxi-consistent set.

Details omitted. (Try them!)

There is a relationship between deducibility in Σ ($\Gamma \vdash_\Sigma A$) and Σ-maxi-consistent sets.

From Lindenbaum's lemma it follows that a formula A is deducible from a set of formulas Γ if and only if A belongs to every maximal extension of Γ. And a formula A is a theorem of Σ(i.e. $\vdash_\Sigma A$) if and only if A is a member of every Σ-maxi-consistent set. In other words:

Theorem 8 *[Chellas Thm 2.20, p57]*

(1) $\Gamma \vdash_\Sigma A$ *iff* $A \in \Delta$ *for every* Σ*-maxi-consistent* Δ *such that* $\Gamma \subseteq \Delta$.

(2) $\vdash_\Sigma A$ *iff* $A \in \Delta$ *for every* Σ*-maxi-consistent* Δ.

Proof Exercise. (In the tutorial exercises.)

Proof sets

Definition 9 (Proof set) *The* proof set *of a formula A in system Σ — denoted $|A|_\Sigma$ — is the set of Σ-maxi-consistent sets that contain A.*

In other words, where Γ is a Σ-maxi-consistent set, $\Gamma \in |A|_\Sigma \Leftrightarrow A \in \Gamma$.
Notice that the set of all Σ-maxi-consistent sets is $|\top|_\Sigma$.

This extra notation is quite useful when we look at canonical models (next). But if you don't like it you can ignore it.