Tutorial Exercises 2 (mjs)
(Maxi-consistent sets)

1. *From 2002 exam* S4 is the normal modal logic $KT4$. Prove that if $\{\Box A_1, \ldots, \Box A_n, \neg\Box B\}$ is S4-consistent then so is $\{\Box A_1, \ldots, \Box A_n, \neg B\}$.

   (This is not a question about maxi-consistent sets.)

2. (This is one of the unproved theorems in the notes.) Prove that:

   (a) $\Gamma \vdash_\Sigma A$ iff $A \in \Delta$ for every Σ-maxi-consistent Δ such that $\Gamma \subseteq \Delta$.

   (b) $\vdash_\Sigma A$ iff $A \in \Delta$ for every Σ-maxi-consistent Δ.

   Hint: for the first one, one half is easy, the other half requires Lindenbaum's lemma. The second follows more or less immediately as a special case of the first.

3. (The following result is useful when we define canonical models for normal systems.)

   Prove that for any Σ-maxi-consistent sets Γ and Γ'

$$\{A \mid \Box A \in \Gamma\} \subseteq \Gamma' \quad \Leftrightarrow \quad \{\Diamond A \mid A \in \Gamma'\} \subseteq \Gamma$$

   or equivalently

$$\forall A\,[\,\Box A \in \Gamma \Rightarrow A \in \Gamma'\,] \quad \Leftrightarrow \quad \forall A\,[\,A \in \Gamma' \Rightarrow \Diamond A \in \Gamma\,]$$